Exam 2

14.30 Fall 2004

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Instructions: This exam is closed-book and closed-notes. You may use a simple calculator. Please read through the exam in order to ask clarifying questions and to allocate your time appropriately. You **must show** all your calculations. You have 90 minutes to complete the exam (90 points). Good Luck!

1. **(22 points)**

A. (15 points) Let X and Y be random variables with pdfs $f_X(x)$ and $f_Y(y)$, and joint pdf $f_{X,Y}(x,y)$. Let a, b, and c be constants. Show the following propositions are true. (Hint: For **parts a** and **b** you are expected to use the formal definition of expectation involving integrals and densities; for **part c** you can use, without proving them, any of the results shown in class regarding variances and covariances.)

    a.(4 points) $E(aX + bY + c) = aE(X) + bE(Y) + c$.

    b.(4 points) $E[E(Y|X)] = E(Y)$.

    c.(7 points) If $\text{Cov}(X, Y) = 0$, then $\text{Corr}(X + Y, X - Y) = \dfrac{\text{Var}(X) - \text{Var}(Y)}{\text{Var}(X) + \text{Var}(Y)}$.
(Hint: First show that $\text{Cov}(X + Y, X - Y) = \text{Var}(X) - \text{Var}(Y)$.)

B. (7 points) Let X and Y be discrete random variables. Based on the following joint pmf, compute the variance of X, the expectation of $Z = XY$, and the covariance between X and Y.

$f(x,y)$	$y = 0$	$y = 1$
$x = 0$	.05	.35
$x = 1$	.35	.25

2. (20 points)

a.(9 points) Let $X \sim U[1,3]$. Define a new random variable $Y = -\alpha \ln(3X)$. Using the 1-step method compute the pdf of Y, $f_Y(y)$. (You are expected to show your work in detail, and do not forget to give your answer $\forall y$.)

b.(11 points) Let (X_1, X_2) be a continuous random vector of 2 independent and identically distributed random variables, where

$$f_{X_i}(x) = \frac{1}{\beta}e^{-x/\beta}, \qquad \text{for } 0 < x < \infty \text{ and } \beta > 0.$$

Compute the pdf of the random variable Y, where $Y = \max\{aX_1, X_2 + c\}$, $a > 0$, and $c > 0$. You have the option to answer this question assuming that $c = 0$ and $a > 0$, in which case the maximum score you can obtain is 8 points (instead of 11 points). (You are expected to show your work in detail, and do not forget to give your answer $\forall y$.)

3. (18 points) Assume the following information. The height of adult women in the city of Boston can be characterized with a normal distribution with mean 65 inches and standard deviation 3 inches. The height of adult women in the city of Santiago can be characterized with a normal distribution with with mean 60 inches and standard deviation 2 inches. The total number of women in the city of Boston is 2 million. Finally, Alice lives in Boston and is 66.5 inches tall.

a.(5 points) Find the probability that a randomly chosen woman in the city of Boston is taller than Alice. How many women in the city of Boston are taller than Alice?

b.(5 points) Assume you randomly choose 2 women from the city of Boston and 3 from the city of Santiago. What is the exact probability density function of the sum of their heights?

c.(8 points) How many women from the city of Santiago would you have to pick so that, with at least 95% probability, their average height is less that 1 inch above or below the population mean?

4. (30 points) Copper Inc. produces cathodes of copper. The production process results in a cooper cathode with a mineral law being normally distributed with mean 50 and standard deviation 10. Assume that the production process of each cathode is *iid*.

a.(6 points) A customer of Copper Inc. needs to buy copper cathodes with a mineral law higher than 40 but lower than 60. What is the probability that a particular cathode meets this specification?

b.(6 points) Following **part a**, assume the customer says he is not willing to buy cathodes unless each cathode satisfies his specification with 95% probability. The customer recommends the company to invest in the production process in order to reduce the variability of the cathodes' law. Assuming that the new production process would result in a copper cathode's law being normally distributed with mean 50, how much should the standard deviation of the production process change (from its current value of 10) to satisfy the customer's conditions?

c.(9 points) Following **part c**, assume the company does not change its production process. Assume also that the customer couldn't find another supplier of copper cathodes and that, although he needs only 255, it is thinking on buying 555 cathode from Copper Inc. What is the probability that 255 cathodes out of the total 555 cathodes will satisfy the customer specifications (a law higher than 40 but lower than 60)?

d.(9 points) Copper Inc. can also sell its copper cathodes in the London Market, which are priced according to the law of the mineral: a copper cathode of law L has a price of $\frac{3}{2}L^2$ cents of a dollar. What is the expected price of a copper cathode sold at the London Market?