14.451 Lecture Notes 2

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1 Solving the FE

Now we make more assumptions on the primitives of the problem:

- X is a convex subset of $\mathbf{R}^l$,
- $F(x, y)$ is continuous and bounded,
- Γ is continuous and compact-valued.

Under these assumptions we analyze the functional equation

$$V(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta V(y).$$

Think of the right-hand side of this equation as a map

$$T : C(X) \rightarrow C(X),$$

where $C(X)$ is the space of bounded continuous functions $f : X \rightarrow \mathbf{R}$ with the sup norm. The map is defined as

$$Tf(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y).$$

Crucial observation:

$$f \text{ is a fixed point of } T \iff f \text{ solves } FE.$$

Questions:

- How to show that a fixed point exists?
- Is the fixed point unique?
- How to find a fixed point?

1.1 An example (reaching the center)

Consider the following problem: an agent is located at some point $x_0 \in [-1, 1]$. The agent wants to reach point 0 but traveling is subject to convex costs. Namely, traveling a distance d costs d^2. Moreover, each period the agent pays a cost D^2 for being at a distance D from point 0. The agent discounts payoffs at the rate β.

Let $x_t \in [-1, 1]$ denote the agent location at the beginning of the period. Then the problem is to maximize

$$\sum_{t=0}^{\infty} \beta^t \left(-(x_t - x_{t+1})^2 - x_t^2 \right)$$

subject to

$$\begin{aligned} x_t &\in [-1, 1] \text{ for all } t, \\ & x_0 \text{ given.} \end{aligned}$$

Suppose we focus on functions on $[-1, 1]$ of the following form:

$$V(x) = -Ax^2,$$

for some parameter $A \in \mathbf{R}$. We can restrict attention to $A \geq 0$ because the objective funciton is non-positive.

Now solve

$$\max_{y \in [-1,1]} -(x - y)^2 - x^2 - \beta A y^2$$

first order condition yields

$$y = \frac{1}{1 + \beta A} x$$

and substituting in the objective function yields

$$-\left(1 + \frac{\beta A}{1 + \beta A}\right) x^2.$$

Therefore the Bellman equation becomes

$$-Ax^2 = -\left(1 + \frac{\beta A}{1 + \beta A}\right) x^2.$$

How can we make sure that the function on the left equals the function on the right? Need:

$$A = 1 + \frac{\beta A}{1 + \beta A}.$$

This has a unique solution $A \geq 0$.

We are going to prove it in a way that is much more complicated than necessary, but very useful for what follows.

Define the function $T : \mathbf{R}_+ \to \mathbf{R}_+$ as follows

$$T(A) = 1 + \frac{\beta A}{1 + \beta A}.$$

Now we can prove the following:

Claim 1 *(Contraction) For all A', A''*

$$|T(A'') - T(A')| \leq \beta |A'' - A'|. \tag{1}$$

Proof. Notice that

$$T'(A) = \frac{\beta}{(1 + \beta A)^2} \in [0, \beta] \text{ for all } A.$$

and use the mean value theorem. ■

This allows us to prove.

Claim 2 *If T has fixed point, the fixed point is unique.*

Proof. Suppose there are two fixed points of T, say A' and A'', then

$$|A'' - A'| = |T(A'') - T(A')| \leq \beta |A'' - A'|$$

which gives a contradiction. ■

We also have a way to compute the fixed point A (again much more more complicated than needed, but bear with me...) and thus prove existence.

Start at any $A_0 \geq 0$ and iterate:

$$A_n = 1 + \frac{\beta A_{n-1}}{1 + \beta A_{n-1}}.$$

Now from (1) we have

$$|A_n - A_{n-1}| \leq \beta |A_{n-1} - A_{n-2}| \tag{2}$$

which implies that:

Claim 3 *A_n is a Cauchy sequence, so $\lim_{n\to\infty} A_n$ exists.*

Proof. For any $m > n$

$$\begin{aligned} |A_m - A_n| &\leq |A_m - A_{m-1}| + ... + |A_{n+1} - A_n| \leq \\ &\leq \left(1 + \beta + ... + \beta^{m-n-1}\right) |A_{n+1} - A_n| \leq \\ &\leq (1 - \beta)^{-1} |A_{n+1} - A_n| \leq (1 - \beta)^{-1} \beta^n |A_1 - A_0|. \end{aligned}$$

The first follows from triangle inequality. The second from applying (2) iteratively on each term. The third from

$$1 + \beta + ... + \beta^{m-n-1} < \sum_{j=0}^{\infty} \beta^j = (1 - \beta)^{-1}.$$

The fourth from iterating on (2). So by choosing n we can make sure that $|A_m - A_n| < \varepsilon$ for all $m \geq n$. ■

This implies that A_n converges to some A.

Claim 4 *If* $A = \lim_{n\to\infty}$ *then* A *is a fixed point of* T.

Proof. Notice that

$$\begin{aligned} |T(A) - A| &\leq |T(A) - A_n| + |A - A_n| = \\ &= |T(A) - T(A_{n-1})| + |A - A_n| \leq \beta |A - A_{m-1}| + |A - A_m| \end{aligned}$$

where the first follows from the triangle inequality, the second from the definition of the sequence $\{A_n\}$, the third from (2). Taking the limit as $m \to \infty$ on the last expression we get $|T(A) - A| = 0$, which implies $T(A) = A$. ■

Summing up, using property (1), we have been able to:

- establish existence and uniqueness of solution;
- find a way of computing the solution.

Now we will see how to apply this idea to more general problems, where instead of dealing with a one parameter family of functions on X, we are dealing with a much larger set of functions, in particular the set of bounded continuous functions $C(X)$.

Notice that the set of functions we looked at was a subset of $C([-1, 1])$. Moreover, if $f_A(x) = -Ax^2$ and $f_B(x) = -Bx^2$ then

$$\|f_A - f_B\| = \sup_{x\in[-1,1]} |f_A(x) - f_B(x)| = |A - B|.$$

To find a fixed point we used the map T to search around the space $\mathbf{R}_+$ (which was indexing our space of functions), trying to make the distance between each candidate function and the next smaller and smaller. That is, making $\|f_{n+1} - f_n\| \to 0$. The same strategy can be adopted in general as long as we are able to establish the analog of (1).

1.2 Applying the contraction mapping theorem

Define the distance between two functions $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ as

$$\|f(x) - g(x)\| = \sup_{x\in X} |f(x) - g(x)|.$$

This is what it means to "use the sup norm" to compute the distance between functions.

Consider the space

$$C(X) = \{f : X \to \mathbf{R}, f \text{ is continuous on } X \text{ and } \|f\| < \infty\}$$

Now we want to search for a solution to FE in this space by applying repeatedly the map $T : C(X) \to C(X)$ (as we did in the example) where

$$Tf(x) = \max_{y\in\Gamma(x)} F(x, y) + \beta f(y).$$

What do we need:

1. show that indeed T maps $C(X)$ into $C(X)$;
2. show that some version of condition (1) applies, i.e., that T is a contraction;
3. show that if T is a contraction we can use it to generate a Cauchy sequence of functions $\{f_n\}$ in $C(X)$ (starting at any f_0);
4. make sure that this sequence converges to a function f in $C(X)$.

For 1 we can use the theorem of the maximum (SLP: Theorem 3.6) and our assumptions that F is continuous and that Γ is continuous and compact-valued.

For 2 we use Blackwell's sufficient conditions (SLP: Theorem 3.3).

For 3 we can use the contraction mapping theorem (SLP: Theorem 3.2).

For 4 we use the fact that $C(X)$ is a complete metric space.

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