

14.451: Introduction to Economic Growth

Problem Set 3

Due date: March 9, 2007.

Question 1: Value Function Iteration.

The social planner solves the following problem:

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} U_0 &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad c_t + k_{t+1} &\leq (1 - \tau) f(k_t) + (1 - \delta) k_t \quad \forall t = 0, 1, 2, \dots \\ c_t &\geq 0, k_{t+1} \geq 0 \quad \forall t = 0, 1, 2, \dots \\ k_0 &\geq 0 \text{ given,} \end{aligned}$$

where U_0 is the lifetime utility of the representative agent, k_t is physical capital at time t , and c_t is consumption at time t . Population is normalized to 1. The difference to the standard case is that the government levies taxes on output at tax rate τ . Of course, it throws all the tax revenues into the Atlantic Ocean.

1. Express the above problem as a dynamic programming problem, i.e. write out the Bellman equation. Use the functional forms: $u(c_t) = \ln c_t$ and $f(k_t) = Ak_t^\alpha$. What is the highest level of capital \bar{k} that can be feasibly sustained in this economy?
2. Derive the Euler equation for consumption and the transversality condition (assuming the value function is differentiable). Find an analytical expression for the steady state value of the capital stock.
3. Write a program in Matlab to implement the calculation and visual depiction of the policy functions for this problem. You should use the code that was covered in recitation as a guide (the code for growthDP.m, along with auxiliary functions u.m and f.m, are posted on the website). Instructions are as follows:
 - (i) Use Matlab to calculate the numerical value of the steady state levels of the capital stock, k_{SS} , and consumption, c_{SS} . Input the following

parameter values:

$$\begin{aligned} A &= 1 \\ \alpha &= 0.33 \\ \beta &= 0.97 \\ \delta &= 0.1 \\ \tau &= 0.3 \end{aligned}$$

- (ii) Define a grid for capital k with 1000 values. Let the capital values in the grid be between 0.01 to 1.2 times the steady state level.
 - (iii) Run a value function iteration. Explain how you can use the Bellman operator you derived in part 1 in order to generate a sequence of guesses for the value function. Start with any initial guess for the value function and generate this sequence. Iterate until the value function has converged. (In practice, stop the iteration when the difference between your old guess and your new guess is sufficiently small).
4. Plot your value function $V(k)$ and the policy function for next period's capital $k'(k)$. Also plot the graphs for (i) output net of taxes and depreciation $(1 - \tau) f(k) + (1 - \delta) k$, and (ii) consumption $c(k)$ on the same axes. Submit these graphs along with your Matlab code.
 5. Plot the same graphs for $\tau = 0.8$. Compare to the graphs in part 4. Again, submit graphs and code.

Question 2: Forward Shooting Algorithm.

Consider the continuous time version of the Ramsey growth model with wasteful government taxation (as in question 1):

$$\max_{\{c(t), k(t)\}_{t=0}^{\infty}} U_0 = \int_0^{\infty} \exp(-\rho t) u(c(t)) dt$$

$$\begin{aligned} \text{s.t.} \quad \dot{k}(t) &= (1 - \tau) f(k(t)) - \delta k(t) - c(t) \\ k_0 &\geq 0 \text{ given,} \end{aligned}$$

where population is normalized to 1.

1. Use the functional forms: $u(c(t)) = \frac{c(t)^{1-\frac{1}{\theta}}}{1-\frac{1}{\theta}}$ and $f(k(t)) = Ak(t)^\alpha$. Derive the Euler equation for consumption and the transversality condition. Find an analytical expression for the steady state value of the capital stock in terms of the model parameters.

2. Now we will write a program in Matlab in order to draw the phase diagram of this system. The phase diagram summarizes the dynamics given by the Euler equation and the capital accumulation equation. In order to incorporate the transversality condition, we need to impose the restriction that the generated sequence of consumption and capital converges to the steady state discussed in lectures. I will provide detailed instructions on how to do this. There are 4 files to construct.

shoot.m

- (i) Set up a Matlab file called shoot.m. Use Matlab to calculate the numerical value of the steady state levels of the capital stock, k_{SS} , and consumption, c_{SS} . Input the following parameter values:

$$\begin{aligned} A &= 1 \\ \alpha &= 0.33 \\ \rho &= 0.03 \\ \delta &= 0.1 \\ \theta &= 2 \\ \tau &= 0.3 \end{aligned}$$

- (ii) Set $k(0) = k_0$ equal to half the value of the steady state capital stock. The problem is to find the value of $c(0)$ that is consistent with the Euler equation, capital accumulation equation and the transversality condition. We will search between $c_L = 0$ and c_H given by the $\dot{k} = 0$ locus. Define these values in your code.
- (iii) Set up a loop in order to generate consecutive candidate paths for consumption and capital. We will define c_L and c_H anew for each iteration. Algorithm:

- Step 1: Given these values, set the guess $c_0 = \frac{1}{2}(c_L + c_H)$.
- Step 2: Use the ode45 function in Matlab to generate a sample path from the initial condition (k_0, c_0) . Read the help file and look at the examples.

Use the format $[t, y, TE, YE, IE]$ on the left hand side. t is time; y is a 1×2 vector of the state variables, k and c . On the right hand side, follow these instructions. Set the odefun to be @ramsey (i.e., define the differential equation system in the function **ramsey.m**). Set the tspan to run from 0 to 5000. Set the initial conditions to be $[k_0 \ c_0]$.

Now consider the options settings (read the help file for odeset for this). Set the installable output function to be odephas2 for two-dimensional phase plane plotting. Also use the event location property but don't use their file for this. Set up your own, called **events.m**. I will describe below how we can use this function to stop the generation of the sample path whenever \dot{c} or \dot{k} are negative. Let the last iteration be at time T .

- Step 3: If $|c(T) - c_{SS}| < 10^{-4}$, stop the loop. Then check whether the generation of the sample path was stopped because $\dot{c} < 0$ or because $\dot{k} < 0$. This information will tell us whether the initial guess for c_0 was too high or too low. If too high, reset $c_H = c_0$. If too low, reset $c_L = c_0$. Then go back to step 1.

f.m

- (i) Write the production function file as usual.

ramsey.m

- (i) In the first line of the file, set the function ramsey to be a function of two variables (t, y) . To repeat: t is time; y is a 1×2 vector of the state variables, k and c .
- (ii) Set the output of this file to be a 2×1 vector dy , defined using the elements of the vector y . The first component of this vector is the expression for \dot{k} . The second component is the expression for \dot{c} .

events.m

- (i) Instructions for setting up the events.m is contained in the help file for ode45. To start with, set the function events to be a function of two variables (t, y) .
 - (ii) Set up two event functions. We do this by setting $\text{value} = [\dot{k}_t, \dot{c}_t]$. Define each of the components of this vector using the expressions for \dot{k} and \dot{c} (you will need to do this before you define value).
 - (iii) Set the integration (i.e. generation of the sample path) to end if either of the event functions reaches a zero.
 - (iv) Allow all the zeros of the event functions to be computed (the default).
 - (v) Following the instructions above will mean that corresponding entries in TE , YE , and IE (of the file **shoot.m**) return, respectively, the time at which an event occurs (this is T), the solution for y at the time of the event, and the index i of the event function that vanishes.
3. Congratulations! You have generated the stable arm from below. Now generate the stable arm from above (set $k(0) = k_0$ equal to one and a half times the value of the steady state capital stock). Plot the phase diagram in Matlab with the two branches of the stable arm, for values of capital between $\frac{1}{2}$ and $1\frac{1}{2}$ times the steady state value. Submit this plot of the phase diagram together with your Matlab code.

Bonus: Use the expression for $\frac{dc(t)}{dk(t)}$ and l'Hôpital's Rule to calculate $\frac{dc}{dk}|_{(k_{SS}, c_{SS})}$. There will be 2 roots. Use this information to plot the unstable arms in your phase diagram.