14.452 Recitation #2

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November 2016

Logistics

- Class today + tomorrow, recitation next Tuesday
- Problem set solutions online
- Piazza

Plan for today

1. A general approach to Solow models
    - Problem 4
    - Problem 3
2. Local stability analysis of ODEs
    - Neoclassical Growth Model

Section 1

A general approach to Solow models

Solow model

- $f(K,t) =$ **output** at time t given capital K
  - $f(K,t)$ weakly concave & strictly increasing in $K \geq 0$
- **Solow model:**

$$\dot{K} = sf(K,t) - \delta K$$

  where $s > 0$, $K_0 > 0$.

- **Steady state equilibrium (BGP):** K grows at rate $g \in \mathbb{R}$
- **Asymptotic BGP,** if

$$\lim_{t\to\infty} \frac{\dot{K}}{K} = g$$

  or more precisely: $e^{-gt} K \to const$

Two questions

1. Does a BGP exist? If so, what K_0 does it require?
2. Does an asymptotic BGP exist? If so, what K_0 does it require?

Examples from class

- **Population growth:** $f(K,t) = F(K, L(t))$ (with CRS F)
  - Q1: Yes, if $K_0 = L_0 k^*$. Q2: Yes, for any $K_0 > 0$.
- **Harrod-neutral techn. change:** $f(K,t) = F(K, A(t)L)$
  - Q1, Q2: same
- AK **technology:** $f(K,t) = AK$
  - Q1,Q2: Yes, for any $K_0 > 0$.

Examples from the problem set

- **Problem 1:** $f(K,t) = L(t)^{\beta} K^{\alpha} Z^{1-\alpha-\beta}$
- **Problem 3:**
$$f(K,t) = \left(\gamma\left(A_K(t)K\right)^{(\sigma-1)/\sigma} + (1-\gamma)\left(A_L(t)L\right)^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)}$$
- **Problem 4:** $f(K,t) = q(t)F(K,L)$

$\rightarrow$ **Today: answer Q1 and Q2 for Problems 3 & 4**

Idea for solving general Solow model

- Guess growth rate g
- Define $k(t) \equiv K(t)/e^{gt}$
    - alternative: divide by variable proportional to e^{gt} (e.g. labor, techn.)
- Gives ODE

$$\dot{k} = se^{-gt}f(ke^{gt}, t) - (\delta + g)k$$

- **Idea:** Study **limiting ODE**

$$\dot{k} = s\hat{f}(k) - (\delta + g)k$$

- Note: Limiting ODE = ODE in examples from class

Three steps

1. Find **growth rate** g s.t. $e^{-gt} f(ke^{gt}, t) \to \hat{f}(k)$ finite and positive
2. Find **steady states** k^* of **limiting ODE**

$$\dot{k} = s\hat{f}(k) - (\delta + g)k \tag{1}$$

3. Get answers

(Q1) If **"no limit condition"** holds

$$e^{-gt} f(k^* e^{gt}, t) = \hat{f}(k^*) \text{ for all } t$$

$\Rightarrow$ BGP exists for $K_0 = k^*$

(Q2) If k^* globally stable: asymptotic BGP exists for any $K_0 > 0$

Subsection 1

Problem 4

Setup

- Production function

$$f(K,t) = q(t)F(K,L)$$

$$q(t) = e^{\gamma_K t}$$

- Ask Q1 & Q2
- Two cases:
  1. $F = K^{\alpha} L^{1-\alpha}$
  2. Any kind of F

Cobb-Douglas F

1. **Growth rate** g such that

$$e^{-gt}f(ke^{gt},t) = e^{-(g-\gamma_K)t}F(e^{gt}k,L) \to \text{finite \& positive}$$

   - Here

$$e^{-(g-\gamma_K)t}F(e^{gt}k,L) = e^{(\alpha g - g + \gamma_K)t}k^{\alpha}L^{1-\alpha}$$

   - Finite and positive precisely if $g = \frac{\gamma_K}{1-\alpha}$
   - **no limit condition** holds for any k

2. **Limiting ODE:**

$$\dot{k} = sk^{\alpha}L^{1-\alpha} - (\delta+g)k$$

   has **globally stable steady state**

$$k^* = \left(\frac{s}{\delta+g}\right)^{1/(1-\alpha)}L$$

3. **Q1:** Yes if $K_0 = k^*$. **Q2:** Yes for any $K_0 > 0$.

General F: BGP?

- If there is a BGP, say with $K_0 = k^*$, then **no limit condition** holds

$$e^{-(g-\gamma_K)t}F(e^{gt}k^*, L) = \hat{f}(k^*) \in (0, \infty)$$

  at all times t

- Define $x \equiv e^{gt}k^*$. Thus,

$$F(x, L) = const \cdot x^{\frac{g-\gamma_K}{g}}$$

  for x greater than some lower bound. Basically Cobb-Douglas...

- Hence **no BGP possible** unless exactly Cobb-Douglas for large K!

General F: asymptotic BGP?

- Turns out: Asymptotic BGP still works if F is *asymptotically* Cobb-Douglas, i.e.

$$\frac{d \log F(K,L)}{d \log K} \to \alpha, \text{ as } K \to \infty$$

sufficiently fast (e.g. satisfied by any CES)

General F: asymptotic BGP?

1. **Growth rate** $g = \frac{\gamma \kappa}{1-\alpha}$

$$e^{-gt} f\left(ke^{gt}, t\right) \to \underbrace{const}_{\equiv A} \times k^{\alpha}$$

2. **Limiting ODE:**

$$\dot{k} = sAk^{\alpha} - (\delta + g)k$$

which has **globally stable steady state**

$$k^* = \left(\frac{sA}{\delta + g}\right)^{1/(1-\alpha)}$$

3. **Q1:** No. **Q2:** Yes, for any $K_0 > 0$.

Subsection 2

Problem 3

Setup

- Production function

$$f(K,t) = \left(\gamma\left(A_K(t)K\right)^{(\sigma-1)/\sigma} + (1-\gamma)\left(A_L(t)L\right)^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)}$$

where

$$A_K(t) = e^{g_K t} \text{ and } A_L(t) = e^{g_L t}$$

- $g_K > 0$, $\sigma < 1$
- Ask Q1 and Q2
- Share of labor in national income?

Asymptotic behavior

1. **Growth rate** $g = g_L$ (given). Then:

$$e^{-gt} f(ke^{gt}, t) = \left(\gamma \left(A_K(t)k\right)^{(\sigma-1)/\sigma} + (1-\gamma)\left(A_L(t)Le^{-gt}\right)^{(\sigma-1)/\sigma}\right)^{\sigma}$$

which approaches

$$\hat{f}(k) = (1-\gamma)^{\sigma/(\sigma-1)} L$$

2. **Limiting ODE**

$$\dot{k} = s(1-\gamma)^{\sigma/(\sigma-1)} L - (\delta + g)k$$

has **globally stable steady state**

$$k^* = \frac{s(1-\gamma)^{\sigma/(\sigma-1)} L}{\delta + g}$$

3. **Q1:** No. **Q2:** For any $K_0 > 0$.

Share of labor in national income...

- ... is given by

$$\frac{\frac{\partial Y}{\partial L} L}{Y} = \frac{(1-\gamma)\left(A_L(t)L\right)^{(\sigma-1)/\sigma}}{\gamma\left(A_K(t)K\right)^{(\sigma-1)/\sigma} + (1-\gamma)\left(A_L(t)L\right)^{(\sigma-1)/\sigma}}$$

- Approaches 1 if $g_K > 0$

Section 2

Linearized NGM

Linearizing ODEs

- Idea:

$$\dot{\mathbf{x}} = g(\mathbf{x})$$

  with steady state

$$g(\mathbf{x}^*) = 0$$

- Small deviations from $\mathbf{x}^*$,

$$\tilde{\mathbf{x}} \equiv \mathbf{x} - \mathbf{x}^*$$

  satisfy

$$\dot{\tilde{\mathbf{x}}} \approx J_g^* \cdot \tilde{\mathbf{x}}$$

  where $J_g^* = J_g(\mathbf{x}^*)$ is the Jacobian of g at $\mathbf{x}^*$.

- $\rightarrow$ **Linear ODE system!**

What does the linear system buy us?

- Assume $\mathbf{z}$ is a (real) eigenvector of J_g^* i.e.

$$J_g^* \mathbf{z} = \lambda \mathbf{z}$$

  for some $\lambda \in \mathbb{C}$.

- **Result:** If we start with $\tilde{\mathbf{x}}_0 = \mathbf{z}$, the solution is

$$\tilde{\mathbf{x}}(t) = \mathbf{z} e^{\lambda t}$$

- In particular:
  - $\lambda < 0$: stable along $\mathbf{z}$
  - $\lambda > 0$: unstable along $\mathbf{z}$
  - $Re(\lambda) = 0$: linearization uninformative about local dynamics
- ODE system **saddle path stable** if some λ's are > 0, some are < 0

NGM ODEs...

- ...were

$$\dot{k} = f(k) - (n+\delta)k - c$$

$$\dot{c} = \frac{1}{\epsilon} c \left(f'(k) - \delta - \rho \right)$$

- Steady state:

$$c^* = f(k^*) - (n+\delta)k^*$$

$$f'(k^*) = \delta + \rho$$

Jacobian

- The Jacobian here is

$$J^* = \begin{pmatrix} \partial \dot{k} / \partial k & \partial \dot{k} / \partial c \\ \partial \dot{c} / \partial k & \partial \dot{c} / \partial c \end{pmatrix}$$

- Computing it

$$J^* = \begin{pmatrix} f'(k^*) - (n + \delta) & -1 \\ \frac{c^*}{\epsilon} f''(k^*) & 0 \end{pmatrix}$$

- So the linearized ODE is

$$\begin{pmatrix} \dot{\tilde{k}} \\ \dot{\tilde{c}} \end{pmatrix} = J^* \cdot \begin{pmatrix} \tilde{k} \\ \tilde{c} \end{pmatrix}$$

- What are the eigenvalues?

Eigenvectors

- Characteristic polynomial

$$P(\lambda) \equiv \det\left(J^* - \lambda I\right)$$

$$P(\lambda) = \lambda^2 - \lambda\left(f'(k^*) - (n+\delta)\right) + \frac{c^*}{\epsilon} f''(k^*)$$

- Note: $P(0) < 0$ and therefore two eigenvalues,
  - $\lambda_1 < 0$ stable, with eigenvector (z_1, z_2)
  - $\lambda_2 > 0$ unstable
- Local stable arm: If $\tilde{x}_0 \propto (z_1, z_2)$, then

$$\tilde{x}(t) = \tilde{x}_0 e^{\lambda_1 t}$$

- Any other $\tilde{x}_0 \not\propto (z_1, z_2)$ has some weight λ_2 eigenvector $\longrightarrow$ unstable!