

14.452. Topic 4. Introducing investment (and saving) decisions

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1. Motivation

In the benchmark model (and the RBC extension), there was a clear consumption/saving decision.

But there was no investment decision. More specifically:

- From the point of view of firms, there was a demand for capital (capital services) every period:

$$Z_t F_K(K_t, N_t) = r_t + \delta$$

The demand for capital was such as to equal to the marginal product of capital to the interest rate plus the discount rate.

- From the point of view of the economy (general equilibrium), the capital stock at t was given from past decisions, and so the same equation determined the equilibrium one-period interest rate at t .

- In other words, the interest rate was always equal to the marginal product of capital.
- In fact, the interest rate appears often to differ from the marginal product of capital.

This suggests that firms face costs of adjusting their capital.

By introducing adjustment costs, we shall:

- Get well-defined consumption and investment demands, which depend on current and expected future interest rates, profit, wages.
- See the role of the term structure of interest rates—the set of intertemporal prices which clears the goods market (investment plus consumption equal production), now and in the future.
- Provide a forward looking version of the IS relation, in which aggregate demand depends on current and expected income and interest rates.
- In the open economy version, get a rich theory of the current account, with a role for factors affecting investment or/and saving.

2. The optimization problem

Consider the following modification of the benchmark optimization problem (i.e. leaving aside the labor/leisure choice):

$$\max E \left[\sum_0^{\infty} \beta^i U(C_{t+i}) | \Omega_t \right]$$

subject to:

$$C_{t+i} = G(K_{t+i}, N_{t+i}, I_{t+i}, Z_{t+i}) \quad N_{t+i} \equiv 1$$

$$K_{t+i+1} = (1 - \delta)K_{t+i} + I_{t+i}$$

The change from the benchmark is the presence of a *net output function*: Gives net output, given inputs K_t and N_t , and investment I_t . So: $G_K > 0, G_N > 0, G_I \leq -1$ (Lucas)

The net output function:

Until now, we assumed that:

$$G(K_t, N_t, I_t, Z_t) \equiv Z_t F(K_t, N_t) - I_t$$

Net output equal to output minus investment. No additional cost involved in investing I_t . Reasonable to assume instead costs of adjustment:

- Once capital is in place, difficult and costly to remove (irreversibility).
- On the other side, a high rate of investment may come with substantial adjustment and installation costs. (Think of building a plant in a month or in a year).

A simple way of capturing this is:

$$G(K_t, N_t, I_t, Z_t) \equiv Z_t F(K_t, N_t) - I_t \left(1 + f\left(\frac{I_t}{K_t}\right)\right)$$

with $f(\cdot)$ increasing in I/K .

$$G(K_t, N_t, I_t, Z_t) \equiv Z_t F(K_t, N_t) - I_t \left(1 + f\left(\frac{I_t}{K_t}\right)\right)$$

- Can capture irreversibility. For example: $f = 0$ for $I > 0$, $f = -1$ if $I < 0$. (Arrow)
- An easier formulation is to assume that f is linear:

$$G(K_t, N_t, I_t) \equiv Z_t F(K_t, N_t) - I_t \left(1 + a \frac{I_t}{K_t}\right)$$

This is what we use below.

Cost of installation per unit a function of the ratio of investment to capital?
To maintain constant returns to scale. If $F(., .)$ itself has CRS, then:

$$G(\lambda K_t, \lambda N_t, \lambda I_t, Z_t) = Z_t F(\lambda K_t, \lambda N_t) - \lambda I_t \left(1 + f\left(\frac{\lambda I_t}{\lambda K_t}\right)\right) = \lambda G(K_t, N_t, I_t, Z_t)$$

3. The optimization problem for an open economy

- Start with optimization problem for small open (interest rate taking) economy.
- Shows more clearly the consumption and investment decisions (closer to partial equilibrium)
- Current account adjusts, rather than the term structure of interest rates.
- See BF, 2-4 for continuous time, no uncertainty.
- In closed economy, the term structure must adjust to clear current and future goods markets. Come back to it later.

Economy same as above, but can borrow and lend at a given (gross) rate R (non stochastic). Let A_t be the net asset position of the country in period t . The optimization problem is given by:

$$\max E \left[\sum_0^{\infty} \beta^i U(C_{t+i}) | \Omega_t \right]$$

subject to:

$$A_{t+i+1} = Z_{t+i} F(K_{t+i}, 1) - I_{t+i} \left(1 + a \frac{I_{t+i}}{K_{t+i}} \right) + R A_{t+i} - C_{t+i}$$

$$K_{t+i+1} = (1 - \delta) K_{t+i} + I_{t+i}$$

The economy has two ways of saving for the future: capital at home, and lending to/borrowing from the rest of the world.

4. Deriving the first order conditions

Let the Lagrange multipliers associated with the first constraint be $\beta^i \lambda_{t+i}$ and that associated with the second constraint be: $\beta^i \mu_{t+i}$.

Write the Lagrangian and take derivatives.

The first two equations characterize the **behavior of consumption**.

$$C_t : \quad U'(C_t) = \lambda_t$$

$$A_{t+1} : \quad \lambda_t = E[\beta R \lambda_{t+1} \mid \Omega_t]$$

- Marginal utility of consumption has to equal the marginal utility of wealth.
- The marginal utility of wealth today is equal to the expected marginal utility of wealth tomorrow times the rate of return on bonds, discounted by the discount factor.

If $\beta R = 1$ (a condition needed, if the rate is constant, to have a stationary state), then:

$$U'(C_t) = E[U'(C_{t+1} | \Omega_t)]$$

The next two equations describe the **behavior of investment**:

$$I_t : \quad \lambda_t \left(1 + 2a \frac{I_t}{K_t}\right) = \mu_t$$

$$K_{t+1} : \quad \mu_t = \beta E \left[\lambda_{t+1} \left(Z_{t+1} F_K(K_{t+1}, 1) + a \left(\frac{I_{t+1}}{K_{t+1}} \right)^2 \right) + (1 - \delta) \mu_{t+1} \mid \Omega_t \right]$$

- MC of investing $(1 + 2a(I/K))$, times the MU consumption (λ_t) , must be equal to the marginal value of installed capital (μ_t) .
- MV of installed capital this period is equal to the expected value of MPK next period times the MU of consumption next period, plus the expected discounted MV of installed capital next time, adjusted for depreciation.
Note MPK has two terms: The direct MPK, and the marginal decrease in installation cost.

It is useful to define a new variable

$$q_t \equiv \frac{\mu_t}{\lambda_t}$$

Interpretation: q_t as the marginal value of capital in place in terms of goods.
Then, we can rewrite the two first order conditions as:

$$\frac{I_t}{K_t} = \frac{1}{2a}(q_t - 1)$$

$$q_t = E \left[\beta \frac{U'(C_{t+1})}{U'(C_t)} \left\{ (Z_{t+1} F_K(K_{t+1}, 1) + a \left(\frac{I_{t+1}}{K_{t+1}} \right)^2) + (1 - \delta) q_{t+1} \right\} \mid \Omega_t \right]$$

(What happens as a goes to zero (no adjustment cost)?

Simple characterization of investment behavior (the “q” theory):

- Investment proceeds until the marginal cost of investment is equal to the marginal value of capital in place.

Rate of investment is an increasing function of the shadow marginal value of capital, of **marginal q** for short. If for example $q_t = 1$, then the optimal rate of investment is zero:

- Marginal q is in turn equal to the expected present value of the marginal product of capital. Solve recursively to get:

$$q_t = E \left[\sum_{i=1}^{\infty} (1 - \delta)^{i-1} \beta^i \frac{U'(C_{t+i})}{U'(C_t)} \left\{ (Z_{t+i} F_K(K_{t+i}, 1) + a \left(\frac{I_{t+i}}{K_{t+i}} \right)^2) \mid \Omega_t \right\} \right]$$

where I have used the fact that

$$\beta \frac{U'(C_{t+i})}{U'(C_{t+i-1})} \beta \frac{U'(C_{t+i-1})}{U'(C_{t+i-2})} \dots \beta \frac{U'(C_{t+1})}{U'(C_t)} = \beta^i \frac{U'(C_{t+i})}{U'(C_t)}$$

Note the system is not recursive. I_t depends on future MPKs which depend on future Ks, which depend on investment today.

Can solve it through linearization. First combine the last two FOCs and the accumulation equation to eliminate I :

$$q_t = E\left[\beta \frac{U'(C_{t+1})}{U'(C_t)} \left\{ (Z_{t+1} F_K(K_{t+1}, 1) + \frac{1}{4a} (q_{t+1} - 1)^2) + (1 - \delta)q_{t+1} \right\} \mid \Omega_t \right]$$

$$K_{t+1} = \left(1 - \delta + \frac{1}{2a}(q_t - 1)\right)K_t$$

If we linearize, then we can replace the linear approximation to the marginal rate of substitution by the riskless rate. This gives a system of two equations in q_t, K_t , to be solved in standard fashion.

To summarize: We have derived a (relatively) simple characterization of consumption and investment behavior:

- Investment depends on marginal q , which depends on current and future marginal products of capital, so on current and expected technological shocks.

Note that the investment decision does depend on the marginal rate of substitution of consumers;

This effect disappears in the linear (or log linear) approximation, and we get a clean separation between investment and consumption decisions.

Firms solve for optimal investment. Then consumers choose consumption.

- Consumption depends on current and future income, net of investment spending. Consumers tilt and smooth in the usual way.

5. Consumption, investment, and the current account, in the open economy

From the FOC, we can guess the effects of a favorable technological shock on consumption, investment, and the current account.

- At the earlier path of capital, marginal q goes up. Investment increases. The more permanent the shock, the larger the increase.
- Consumption goes up as well. No tilting effect here, as R is not affected. So anticipations of higher output net of investment spending lead to higher consumption.
- So higher investment, higher consumption. Higher output as well. If permanent shock, initial current account deficit. (How can we be sure? Suppose costs of adj infinite, so I does not change.)
- Accumulation of debt over time. (Steady state level history dependent)

6. Consumption, investment, and the term structure of interest rates, in the closed economy

In a closed economy, investment plus consumption must equal to output.

Equivalently: Current and future interest rates have to generate a path of consumption and investment such that the goods market clears and is expected to clear.

(In open economy, interest rate and term structure are given. Current account is what adjusts)

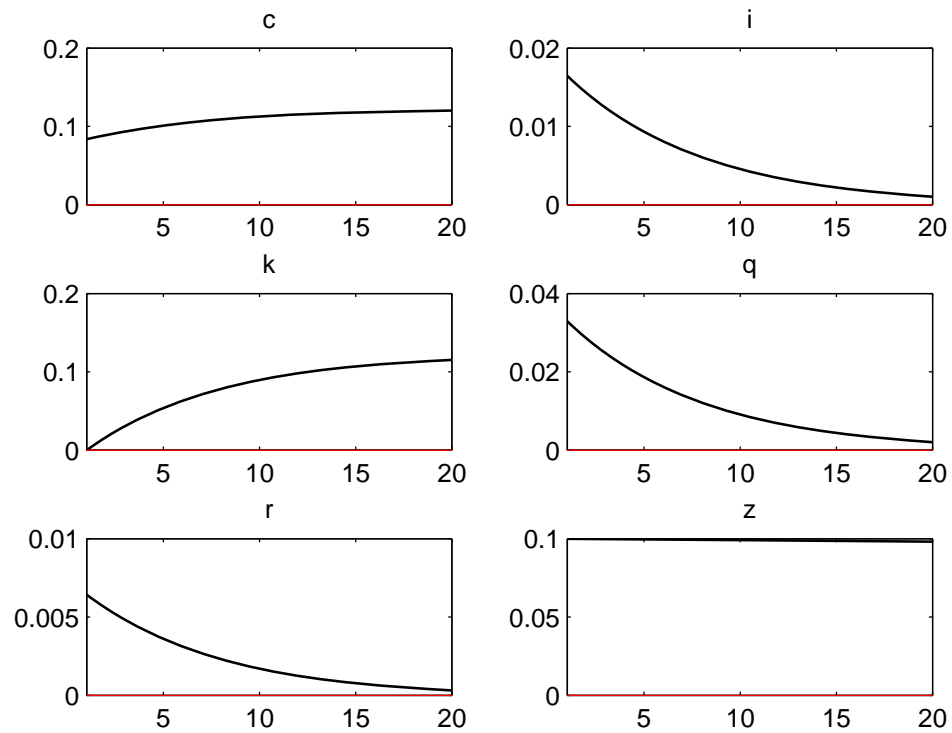
The effects of a permanent, unexpected, productivity shock

- An initial increase in investment. (Same steady state as Ramsey model, so higher capital in steady state, higher depreciation, higher replacement investment)
- Initial increase or decrease in consumption. Eventually higher.
- Marginal value of capital in place goes up. (Later, see close relation with stock market valuation)
- Interest rate. Up initially, less than MPK, and then returns to modified golden rule level.
- Term structure: downward sloping.

Dynare simulation.

Parameters:

- log consumption, and Cobb Douglas production
- $\alpha = 0.2$
- $\delta = 0.0$ (why?)
- $\beta = 1/1.2$ (why?)
- $Z = \rho Z_{t-1} + \epsilon_t$
- $a = 1.0$



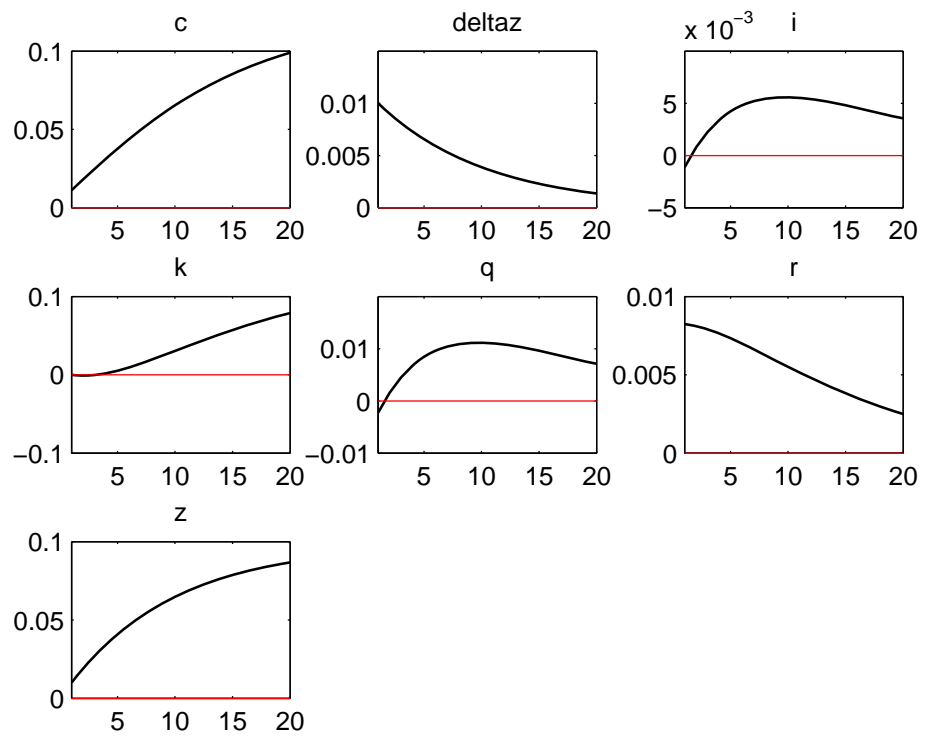
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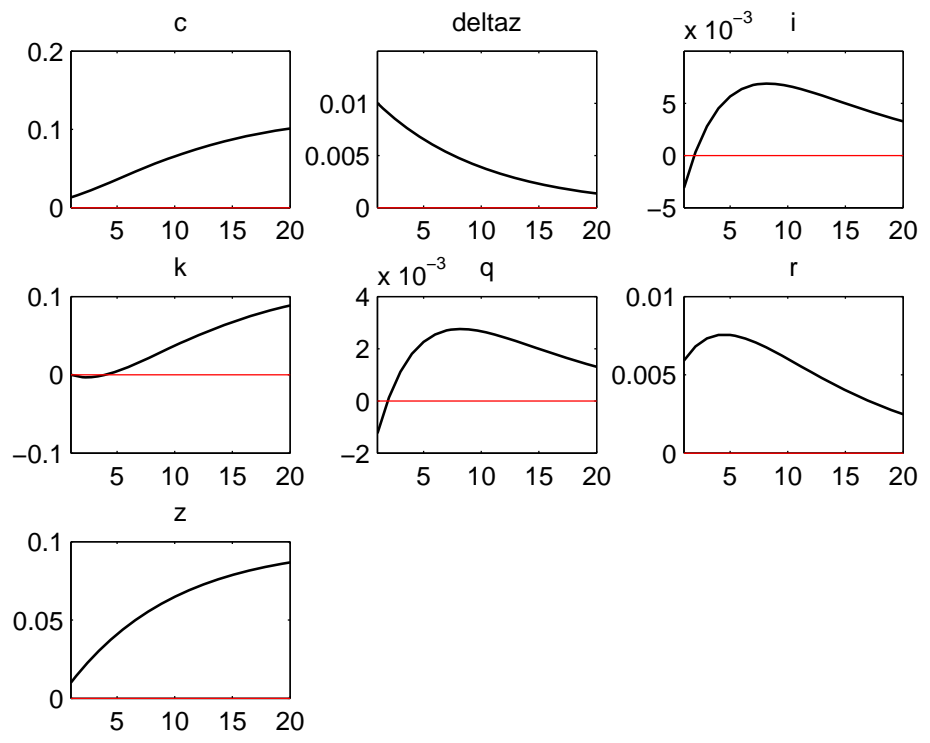
More complex/interesting. The effects of a permanent, expected productivity shock (anticipated to take place in the future).

Say, anticipations in early 1990s that the underlying rate of TFP growth has increased.

- At the previous sequence of interest rates, both consumption and investment go up.
- This cannot be, as output is initially unchanged. Whether it is consumption or investment which goes up is ambiguous and depends on
The elasticity of substitution of consumers: The lower, the more likely consumption will go up.
Adjustment costs. The higher, the more likely investment will go up.
- What sequence of interest rates will achieve this outcome?

Assume: $\Delta Z_t = .9\Delta Z_{t-1} + \epsilon_t$, and $a = 1.0, 5.0$.





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Take up three related issues.

- Aggregate demand and the term structure.
- Marginal q and the stock market.
- The stock market, bubbles, and investment.

7. Aggregate demand, and the term structure.

- Now have a well defined **aggregate demand relation**. c and i functions, if only implicitly. Could introduce g and τ .
- Equilibrium condition: Aggregate demand equals aggregate supply (here given at t given capital and constant employment).
- The role of the term structure of interest rates (yield curve) in achieving equilibrium.
- What if interest rates do not adjust in this way? Suppose that they are instead determined by monetary policy.
- Then, aggregate demand not equal to aggregate supply. If supply adjusts (big if), then anything that affects aggregate demand will affect the equilibrium.
- Long way, but will get there: (1) Nominal rigidities, and a role for monetary policy to affect interest rates.(2) Price setters willing to accomodate demand.

8. Marginal and average q , and investment

Consider:

- The (unobservable) marginal value of capital, “marginal q ”, q_t
- The (observable) average value of capital, “average q ”, “Tobin’s q ”, the ratio of the market value of the capital stock (value of stocks plus value of bonds), divided by the capital stock

Then, under the assumptions we have made, marginal and average q are equal. So a tight relation between the rate of investment and the market valuation of capital.

Think of a firm operating in this economy. Its value (after profit has been paid out this period) is given by:

$$V_t = E\left[\beta \frac{U'(C_{t+1})}{U'(C_t)} (\pi_{t+1} + V_{t+1}) \mid \Omega_t \right]$$

Assume that the firm rents labor but buys and installs capital. Let π_t be the cash flow after paying labor and buying and installing capital, so:

$$\pi_{t+1} = Z_{t+1}F(K_{t+1}, 1) - W_{t+1} - I_{t+1}\left(1 + a\frac{I_{t+1}}{K_{t+1}}\right)$$

We now show that $V_t/K_{t+1} = q_t$.

(Timing is a bit awkward. Result of timing conventions: Firms choose next period's capital stock. In continuous time, the relation reduces to $V/K = q$.)

$$q_t = E\left[\beta \frac{U'(C_{t+1})}{U'(C_t)} \left(Z_{t+1} F_K(K_{t+1}, 1) + a \left(\frac{I_{t+1}}{K_{t+1}} \right)^2 + (1 - \delta) q_{t+1} \right) \mid \Omega_t \right]$$

Multiply both sides by K_{t+1} , to get:

$$q_t K_{t+1} = E\left[\beta \frac{U'(C_{t+1})}{U'(C_t)} \left\{ Z_{t+1} F(K_{t+1}, 1) - W_{t+1} + a \frac{I_{t+1}^2}{K_{t+1}} + (1 - \delta) q_{t+1} K_{t+1} \right\} \mid \Omega_t \right]$$

From the accumulation equation:

$$K_{t+1} = \frac{1}{1 - \delta} [K_{t+2} - I_{t+1}]$$

From the first FOC:

$$q_{t+1} I_{t+1} = \left(1 + 2a \frac{I_{t+1}}{K_{t+1}} \right) I_{t+1}$$

Replace in the equation for $q_t K_{t+1}$ above:

$$q_t K_{t+1} = E\left[\beta \frac{U'(C_{t+1})}{U'(C_t)} \left\{ Z_{t+1} F(K_{t+1}, 1) - W_{t+1} - I_{t+1} \left(1 + a \frac{I_{t+1}}{K_{t+1}}\right) + q_{t+1} K_{t+2} \right\} \mid \Omega_t\right]$$

So,

$$q_t K_{t+1} = V_t$$

Replacing in the first FOC gives a relation between the investment rate and the value of capital in the firm, as assessed by financial markets (between investment and the stock market, for short):

$$\frac{I_t}{K_t} = \frac{1}{2a} \left(\frac{V_t}{K_{t+1}} - 1 \right)$$

$$\frac{I_t}{K_t} = \frac{1}{2a} \left(\frac{V_t}{K_{t+1}} - 1 \right)$$

This relation is not causal. It is an equilibrium relation, which holds when firms maximize their value.

In effect, both asset holders (stock market participants) and firms make the same computation, come to the same conclusions.

This determines both the market value of firms, and the rate of investment.

Note the assumptions needed to yield equality between marginal and average q : (conversely, the reasons why the equality may not hold)

- Constant returns in production.
- Competitive goods markets. No rents.
- A correct measure of capital. Intangibles. (High tech firms?)
- Correct valuation of firms by financial markets. (speculative bubbles?)

How well does it work?

Decently—but no more. (And, in any case, even a tight relation would be limited progress. A relation between two endogenous variables.)

Evidence. I/K and average q over the last fifty years.

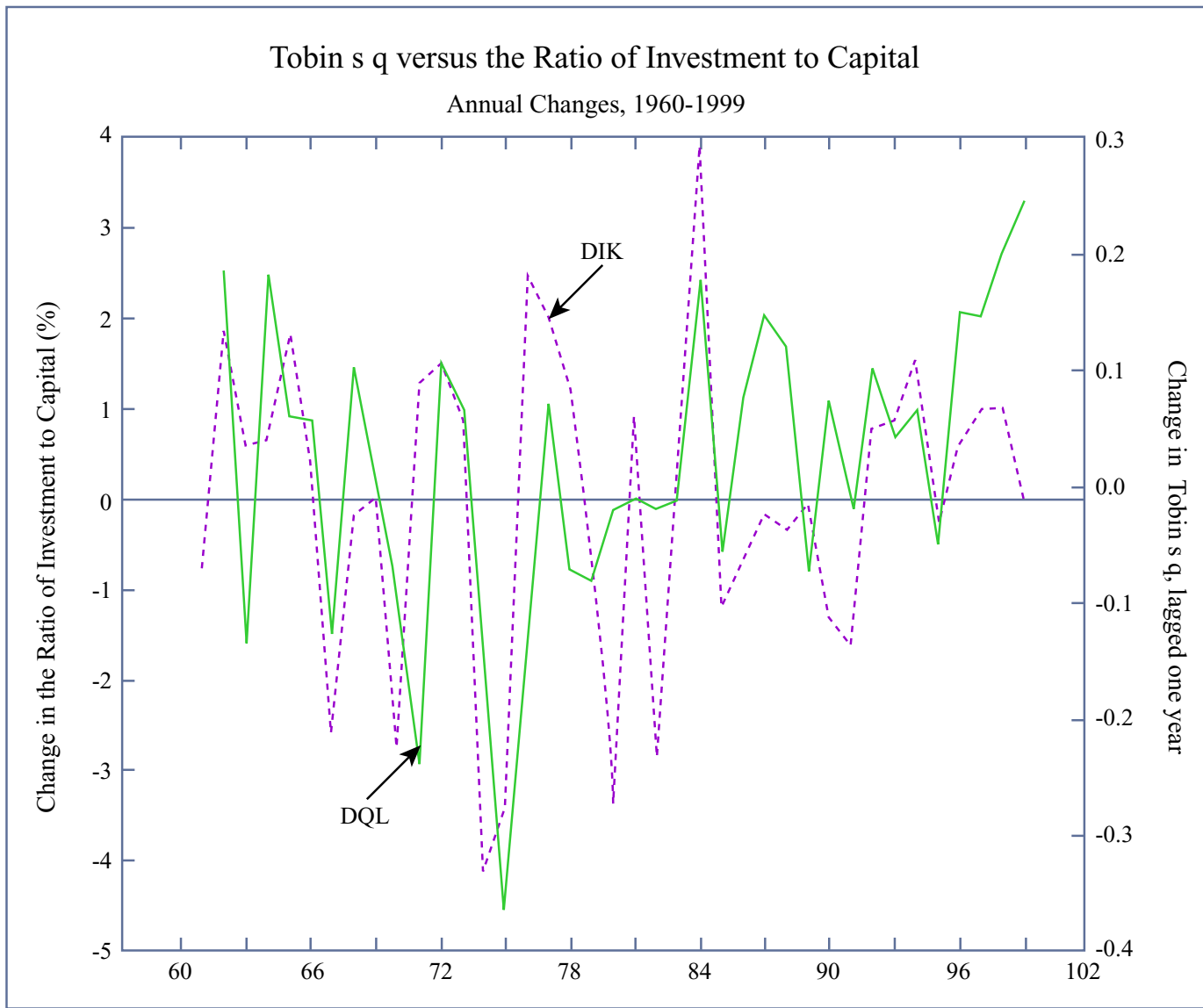


Figure by MIT OCW.

9. Stock market valuation, bubbles, and investment

Back to the arbitrage equation for the value of firms. Assume for notational simplicity that the marginal rate of substitution is constant and equal to $(1 + r)^{-1}$:

$$V_t = E[(1 + r)^{-1}(\pi_{t+1} + V_{t+1})]$$

Solving forward recursively, and assuming $\lim V_{t+j}(1 + r)^{-j} = 0$ gives the solution:

$$V_t^* = E\left[\sum_{j=1}^{\infty} (1 + r)^{-j} \pi_{t+j} \mid \Omega_t\right]$$

The value of the firm is equal to the expected present value of profits. Is this the only solution?

No, if we ignore the transversality condition above, then any solution of the form:

$$V_t = V_t^* + B_t \quad B_t \mid E[B_{t+1} | \Omega_t] = (1 + r)B_t$$

also satisfies the arbitrage equation. We can think of B as a “bubble”, the component of the firm valuation that does not reflect fundamentals.

Examples.

A deterministic bubble:

$$B_{t+j+1} = (1 + r)B_{t+j}$$

A stochastic bubble:

$$B_{t+j+1} = (1 + r)\left(\frac{1}{p}\right)B_{t+j} \text{ with probability } p, 0 \text{ with probability } 1 - p$$

The bubble ends with prob limit 1, but still has an expected value that increases over time at rate $1 + r$.

This raises a large series of questions:

- Can such deviations happen in our general equilibrium models?

The answer (see BF Chapter 5 for more) is no in the infinitely lived models we have looked at. Hits a wall in finite time.

In overlapping generation models, dynamic efficiency is typically enough to rule them out. (Tirole, 1985, EMA): The real value of the bubble component of the asset increases at real rate r , which is then greater than g , the rate of growth of the economy. Eventually bubbles become too large.

If distortions in financial markets, relevant r may be less than g despite the fact that the economy is dynamically efficient.

Can deviations from V^* happen in the real world? The answer appears to be: They can and they do. (See Shiller on reading list).

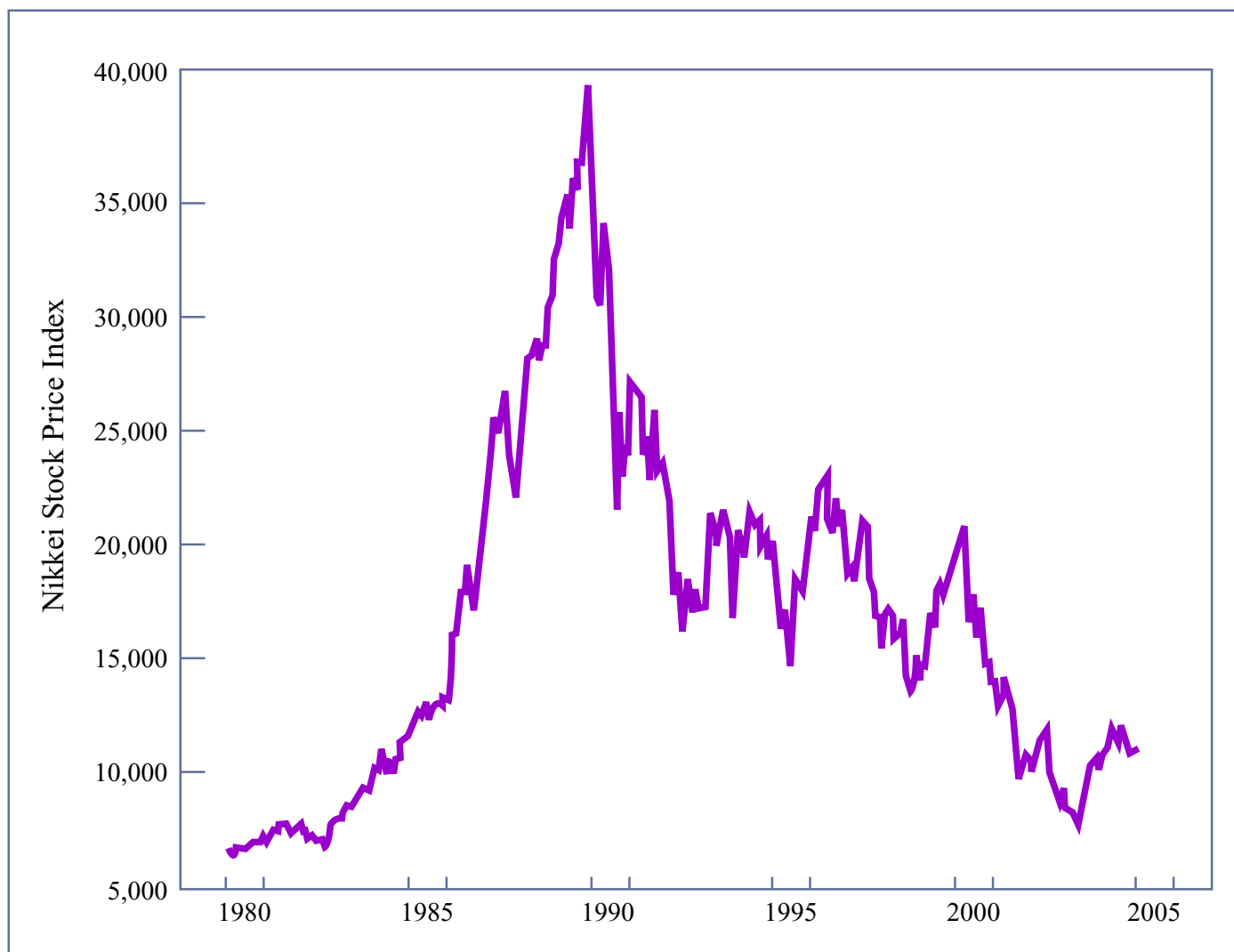


Figure by MIT OCW.

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- Probably do not take the form of "rational" bubbles, but rather of "fads", long lasting stationary deviations from the price.

One can think of those as coming from different sources.

- Noise traders, whose presence is only partly offset by rational, but risk averse arbitrageurs. (DeLong, Shleifer, Summers, Waldman JPE 1990)
- Short-sale constraints in the presence of heterogenous information. As those who would like to be short cannot, the price is higher than it would otherwise be.

- What should firms do (What do firms do) if their estimate of V differs from the market valuation? Ignore the market? Or issue (or buy back) shares? Use the proceeds for investment in the firm, or to buy T-bills?

The answer is unclear. (See Blanchard, Summers, Rhee, QJE 1993, Fahri-Panageas 2004). Depends in part on the source of the deviations.

- What should policy makers, in particular the central bank, do? We shall return to this, but we need first to introduce money, and then a potential role for the central bank.

Summary

- Under costs of adjustment. Investment a function of the marginal q .
- Marginal q equal to the expected present value of marginal profits.
- In an open economy, a permanent technological shock leads initially to an increase in investment, an increase in consumption, and a current account deficit. A transitory shock leads to a current account surplus.
- In a closed economy, the same shock leads initially to an increase in investment, an ambiguous effect on consumption, and an upward shift of the term structure.
- Under CRS and competitive markets, marginal q is equal to average q .