# Solutions for Problem Set #1
# Macro 14.453

September 15, 2006

## 1 Business Cycles Costs

### 1.1 AR(1)

Let utility be given by:

$$E_{-1}\sum_{t=0}^{\infty}\beta^{t}u\left(c_{t}\right)$$

where the instantaneous utility takes the standard CRRA specification

$$u\left(c_{t}\right)=\frac{c_{t}^{1-\gamma}}{1-\gamma}.$$

Assume the following consumption process

$$c_{t}=c_{t-1}^{\alpha}\varepsilon_{t}\exp\left(\mu\right)$$

where

$$\mu=-\frac{\sigma_{\varepsilon}^{2}\left(1-\alpha\right)}{2\left(1-\alpha^{2}\right)}$$

$$\log\varepsilon_{t}\sim N\left(0,\sigma_{\varepsilon}^{2}\right)\quad\forall t$$

so that the logarithm of consumption follows an AR(1) with parameter $\alpha$:

$$\log c_{t}=\mu+\alpha\log c_{t-1}+\log\varepsilon_{t}.$$

1. Some general notation: take a random variable $x$ and define the function $H$ as follows

$$H(x', x) = \Pr\{x_{t+1} \leq x' \mid x_t = x\}.$$

$H$ is called the *transition function*. Then the sequence of distribution functions $\{F_t\}_{t=1}^{\infty}$ for the $x_t$'s is given by

$$F_{t+1}(x') = \int H(x', x)\, dF_t(x) \quad t = 0, 1, 2, ...$$

for a given $F_0$. Then, a distribution function $F$ is called an *invariant distribution* for the transition function $H$ if it satisfies

$$F(x') = \int H(x', x)\, dF(x).$$

In our case $x_t = \log c_t$ and define $z_t = \log \varepsilon_t$, then

$$H(x', x) = \Pr\{\mu + \alpha x_t + z_{t+1} \leq x' \mid x_t = x\}$$

where $x_t$ and $z_{t+1}$ are both normally distributed and independent from each other. We know that the sum of independent normal variables is normally distributed as well with mean equal to the sum of the means and variance equal to the sum of the variances. Then it is clear that the invariant distribution of $x$ will be a normal with mean and variance independent on time, *i.e.* asymptotically

$$x \sim N\left(\mu_c, \sigma_c^2\right)$$

where

$$\mu_c = \mu + \alpha\mu_c \text{ and } \sigma_c^2 = \alpha^2\sigma_c^2 + \sigma_\varepsilon^2$$

so that

$$\mu_c = \frac{\mu}{1-\alpha} \text{ and } \sigma_c^2 = \frac{\sigma_\varepsilon^2}{1-\alpha^2}$$

In fact, we can show that

$$E_t(x_t) = \mu + \alpha E_t(x_{t-1}) = \mu + \alpha\left[\mu + \alpha E_t(x_{t-2})\right] = ...$$

and

$$Var_t(x_t) = \alpha^2 Var_t(x_{t-1}) + \sigma_\varepsilon^2 = \alpha^2\left[\alpha^2 Var_t(x_{t-1}) + \sigma_\varepsilon^2\right] + \sigma_\varepsilon^2 = ...$$

so that, given $x_0 \sim N\left(\frac{\mu}{1-\alpha}, \frac{\sigma_\varepsilon^2}{1-\alpha^2}\right)$, we have

$$\lim_{t\to\infty} E_t(x_t) = \lim_{t\to\infty}\left[\sum_{k=0}^{t-1}\alpha^k\mu + \alpha^t E_t(x_0)\right] = \frac{\mu}{1-\alpha}$$

and

$$\lim_{t\to\infty} Var_t(x_t) = \lim_{t\to\infty}\left[\sum_{k=0}^{t-1}\alpha^{2k}\sigma_\varepsilon^2 + \alpha^{2t} Var_t(x_0)\right] = \frac{\sigma_\varepsilon^2}{1-\alpha^2}.$$

Moreover, note that

$$\begin{aligned} E(c) &= E[\exp(\log c)] = \exp\left(\frac{\mu}{1-\alpha} + \frac{\sigma_\varepsilon^2}{2(1-\alpha^2)}\right) = \\ &= \exp\left(-\frac{\sigma_\varepsilon^2(1-\alpha)}{2(1-\alpha^2)}\frac{1}{1-\alpha} + \frac{\sigma_\varepsilon^2}{2(1-\alpha^2)}\right) = 1 \end{aligned}$$

2. Assuming that the consumption at time 0 ($c_0$) is random and distributed according to the invariant distribution defined in point 1, *i.e.*

$$x_0 \sim N\left(\frac{\mu}{1-\alpha}, \frac{\sigma_\varepsilon^2}{1-\alpha^2}\right)$$

note that you can write

$$V(\lambda, \sigma_\varepsilon^2) \equiv E_{-1}\sum_{t=0}^{\infty}\beta^t u[c_t(1+\lambda)] = E(u[c(1+\lambda)])\sum_{t=0}^{\infty}\beta^t$$

so that

$$\begin{aligned} V(\lambda, \sigma_\varepsilon^2) &= \frac{(1+\lambda)^{1-\gamma}}{(1-\beta)(1-\gamma)}E(c^{1-\gamma}) = \\ &= \frac{(1+\lambda)^{1-\gamma}}{(1-\beta)(1-\gamma)}\exp\left[(1-\gamma)\left(\frac{\mu}{1-\alpha}\right) + \frac{(1-\gamma)^2}{2}\left(\frac{\sigma_\varepsilon^2}{1-\alpha^2}\right)\right] \end{aligned}$$

3. Note that when $\sigma_\varepsilon^2 = 0$, then $\mu = 0$. So we have

$$V(0,0) = \frac{1}{(1-\beta)(1-\gamma)}$$

and we are interested in $\lambda$ such that

$$(1+\lambda)^{1-\gamma} \exp\left[(1-\gamma)\left(\frac{\mu}{1-\alpha}\right) + \frac{(1-\gamma)^2}{2}\left(\frac{\sigma_\varepsilon^2}{1-\alpha^2}\right)\right] = 1$$

Substituting for $\mu$, after some algebra we get

$$(1+\lambda)^{1-\gamma} \exp\left[-\frac{\gamma(1-\gamma)}{2}\left(\frac{\sigma_\varepsilon^2}{1-\alpha^2}\right)\right] = 1$$

and taking log

$$\log(1+\lambda) = \frac{\gamma}{2}\left(\frac{\sigma_\varepsilon^2}{1-\alpha^2}\right)$$

so that

$$\lambda \approx \frac{\gamma}{2}\left(\frac{\sigma_\varepsilon^2}{1-\alpha^2}\right)$$

Note that as the variance or the permanence parameter increase the cost of business cycle increases. This is intuitive since the business cycle is more costly when the associated volatility (and the permanence of the consumption process is in fact a way of increasing its volatility as well) is higher because the risk is higher and the desire of consumption smoothing harder to satisfy. Clearly the more concave is the utility function the higher is the cost associated to volatility.

Recall that in the *i.i.d.* case, *i.e.* $\alpha = 0$ (or no persistency), we have

$$\lambda \approx \frac{\gamma}{2}\sigma_\varepsilon^2$$

and we have the minimum cost associated to the same volatility of the shock $\varepsilon$.

4. Note that when $\alpha \to 1$, the $\log c$ follows a random walk. This means that an invariant distribution for consumption does not exists, as we can simply notice from the fact that if we look for it we find a distribution with infinite variance. Notice that this does not mean that the cost of business cycle (as we defined it) is infinite, but simply that we cannot use the same procedure to evaluate $\lambda$, since we cannot assume an invariant distribution for consumption.

Moreover, let us do a step back. If the consumption is distributed as assumed in this exercise, is it sensible to think at $\lambda$ as the cost of business cycle? Recall that Lucas' calculation are based on the idea that removing the business cycle can be identified with removing the whole stochasticity of the consumption process. Well, if this exercise can be meaningful and insightful when the consumption process is assumed to be *iid*, it is more troublesome to follow the same reasoning when consumption is assumed to have some permanent component, as it seems to be the case in reality. In this case, what we may be interested theoretically would be to disentangle the component of stochasticity that represents "something more" than business cycle from the component of stochasticity that instead creates the oscillations that are typical of the business cycle. Then the correct exercise would be to remove only the latter. In other words low frequency and sometimes very high frequency fluctuations are not regarded as business cycle, as Alvarez and Jermann (2000) point out. Then, it is then that if we remove the whole stochasticity as we do in the exercise, we will find numbers much higher for the cost of business cycle when we have permanent processes for consumption.

5. Suppose now that $c_0$ is known and not random. Note that the previous analysis, based on the assumption that $c_0$ was random and distributed according to the invariant distribution, gave as a sort of upper bound for the cost of business cycle, since we were adding volatility also at time 0, while assuming that consumption at time 0 is known is alleviating the weight of the cost of business cycle.

   Moreover, assume now that

$$u(c_t) = \log(c_t).$$

   I choose to use a recursive formulation, but you can solve the exercise also without it.

   I can rewrite the problem as

$$V\left(c; \lambda, \sigma_\varepsilon^2\right) = u\left[c\left(1+\lambda\right)\right] + \beta E\left[V\left(c'; \lambda, \sigma_\varepsilon^2\right)\right]$$

   and guess

$$V\left(c; \lambda, \sigma_\varepsilon^2\right) = A \log c + B$$

Using this guess I can write

$$A \log c + B = \log [c(1+\lambda)] + \beta E (A \log [c^{\alpha} \varepsilon \exp(\mu)] + B)$$

$$A \log c + B = \log c + \log(1+\lambda) + \beta A \alpha \log c + \beta A E \log \varepsilon + \beta A \mu + \beta B$$

where

$$E \log \varepsilon = 0$$

so that matching the coefficients we get

$$A = \frac{1}{(1-\alpha\beta)}$$

$$B = \frac{\log(1+\lambda)}{(1-\beta)} + \frac{\beta\mu}{(1-\beta)(1-\alpha\beta)}$$

so that we have

$$V\left(c; \lambda, \sigma_{\varepsilon}^{2}\right) = \frac{\log c}{(1-\alpha\beta)} + \frac{1}{(1-\beta)} \log(1+\lambda) + \frac{\beta\mu}{(1-\alpha\beta)(1-\beta)}.$$

Then we are looking for the $\lambda$ such that

$$\log(1+\lambda) = -\frac{\beta\mu}{(1-\alpha\beta)}$$

where

$$\mu = -\frac{\sigma_{\varepsilon}^{2}(1-\alpha)}{2(1-\alpha^{2})}$$

so that

$$\log(1+\lambda) = \frac{\beta(1-\alpha)\sigma_{\varepsilon}^{2}}{2(1-\alpha^{2})(1-\alpha\beta)}$$

or

$$\lambda \approx \frac{\beta\sigma_{\varepsilon}^{2}}{2(1+\alpha)(1-\alpha\beta)}$$

Note that the cost of business cycle is not discontinous in $\alpha = 1$ anymore, since we are not assuming an invariant distribution for consumption to get it. In fact when $\alpha = 0$, we have that

$$\lambda \approx \frac{\beta\sigma_{\varepsilon}^{2}}{2}$$

and when $\alpha = 1$

$$\lambda \approx \frac{\beta\sigma_{\varepsilon}^{2}}{4(1-2\beta)}$$

## 1.2 Random walk

Let the utility be given by:

$$E_{-1}\sum_{t=0}^{\infty}\beta^{t}u\left(c_{t}\right)$$

where the instantaneous utility takes the standard CRRA specification

$$u\left(c_{t}\right)=\frac{c_{t}^{1-\gamma}}{1-\gamma}.$$

Assume the following consumption process

$$c_{t}=c_{t-1}\varepsilon_{t}\exp\left(-\frac{\sigma_{\varepsilon}^{2}}{2}\right)$$

where

$$\log\varepsilon_{t}\sim N\left(0,\sigma_{\varepsilon}^{2}\right)\quad\forall t$$

so that the logarithm of consumption follows a random walk and $E\left(\varepsilon_{t}\right)=1$ $\forall t$.

1. Write the problem in recursive form as follows:

$$V\left(c;\lambda,\sigma_{\varepsilon}^{2}\right)=u\left[c\left(1+\lambda\right)\right]+\beta E\left[V\left(c';\lambda,\sigma_{\varepsilon}^{2}\right)\right]$$

   where $c$ denotes the consumption of today, $c'$ the consumption of tomorrow and $\lambda$ is a positive constant.

   We can guess that the value function assumes the form

$$V\left(c;\lambda,\sigma_{\varepsilon}^{2}\right)=A\frac{c^{1-\gamma}}{1-\gamma}$$

   where $A$ is a constant. Let us verify. By using our guess, we can write

$$A\frac{c^{1-\gamma}}{1-\gamma}=\frac{c^{1-\gamma}\left(1+\lambda\right)^{1-\gamma}}{1-\gamma}+\beta E\left[A\frac{c'^{1-\gamma}}{1-\gamma}\right]$$

   where

$$c'=c\varepsilon\exp\left(-\frac{\sigma_{\varepsilon}^{2}}{2}\right)$$

and

$$\log \varepsilon \sim N\left(0, \sigma_\varepsilon^2\right)$$

So we have

$$A = \frac{(1+\lambda)^{1-\gamma}}{1 - \beta E\left[\varepsilon^{1-\gamma}\right] \exp\left(-\frac{(1-\gamma)\sigma_\varepsilon^2}{2}\right)}$$

Now recall that

$$E\left[\varepsilon^{1-\gamma}\right] = \exp\left[\frac{(1-\gamma)^2 \sigma_\varepsilon^2}{2}\right]$$

so that we can rewrite

$$A = \frac{(1+\lambda)^{1-\gamma}}{1 - \beta \exp\left(-\frac{\gamma(1-\gamma)\sigma_\varepsilon^2}{2}\right)}$$

which is a constant and verifies our guess, so that

$$V\left(c; \lambda, \sigma_\varepsilon^2\right) = \frac{(1+\lambda)^{1-\gamma}}{(1-\gamma)\left[1 - \beta \exp\left(-\frac{\gamma(1-\gamma)\sigma_\varepsilon^2}{2}\right)\right]} c^{1-\gamma}$$

2. Now we want to compute the value of $\lambda$ such that

$$V\left(c; \lambda, \sigma_\varepsilon^2\right) = V(c; 0, 0)$$

or

$$\frac{(1+\lambda)^{1-\gamma}}{(1-\gamma)\left[1 - \beta \exp\left(-\frac{\gamma(1-\gamma)\sigma_\varepsilon^2}{2}\right)\right]} c^{1-\gamma} = \frac{1}{(1-\gamma)(1-\beta)} c^{1-\gamma}$$

We can already see that $\lambda$ turns out to be independent on $c$. In fact we have

$$(1+\lambda)^{1-\gamma} = \frac{\left[1 - \beta \exp\left(-\frac{\gamma(1-\gamma)\sigma_\varepsilon^2}{2}\right)\right]}{(1-\beta)}$$

so that

$$\lambda = \left(\frac{\left[1 - \beta \exp\left(-\frac{\gamma(1-\gamma)\sigma_\varepsilon^2}{2}\right)\right]}{(1-\beta)}\right)^{\frac{1}{1-\gamma}} - 1$$

Then, taking logs

$$\log(1+\lambda) = \frac{\log\left[1-\beta\exp\left(-\frac{\gamma(1-\gamma)\sigma_\varepsilon^2}{2}\right)\right]-\log(1-\beta)}{(1-\gamma)}$$

and approximating we get

$$\lambda \approx \frac{\log\left[1-\beta\exp\left(-\frac{\gamma(1-\gamma)\sigma_\varepsilon^2}{2}\right)\right]-\log(1-\beta)}{(1-\gamma)}$$

Note that $\lambda$ is increasing in $\gamma, \beta$ and $\sigma_\varepsilon^2$ as expected. Note that for $\gamma = 1$ we get exactly the solution we found using directly the log utility function.

3. Looking at the tables we can notice that, in comparison with the *i.i.d.* case, the cost of the business cycle is much higher. This does not necessary invalidates Lucas' point that business cycle is not so costly as slow growth rate, since, as noticed in the previous problem, when the process for consumption is persistent if we evaluate the cost of business cycle as the cost of removing the whole stochasticity of the economy, we are definitely overestimating that cost, since we should disentangle lower frequency randomness from higher frequency one, as discussed in the previous problem. Look at the next table just to get an idea of the magnitude:

| **beta=.9** | | | **gamma** | |
|---|---|---|---|---|
| | | **0.5** | **1.5** | **2** |
| **st.dev.** | **0.01** | 0.022612 | 0.071218 | 0.099447 |
| | **0.05** | 0.115294 | 0.452786 | 0.856803 |

| **beta=.95** | | | **gamma** | |
|---|---|---|---|---|
| | | **0.5** | **1.5** | **2** |
| **st.dev.** | **0.01** | 0.048034 | 0.159651 | 0.236022 |
| | **0.05** | 0.250773 | 1.43844 | 37.68596 |

# 2 Intertemporal Elasticity of Substitution, Risk Aversion and the Cost of Fluctuations

# Solutions Problem Set 1

Macro III (14.453)

First of all a short note on the preferences we have.

$$v_t = \left[(1-\beta)c_t^{\rho} + \beta(E_t v_{t+1}^{\alpha})^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}$$

Note that in that case we no longer restrict (as in the usual framework) the coefficient of risk aversion to be inversely related to the elasticity of intertemporal substitution. Now (1-$\rho$)$^{-1}$ is the elasticity of intertemporal substitution while $\alpha$ captures risk aversion. This will be important when discussing the results of the problem set.

1. In this question you are asked to proof that if $\{v_t\}$ is implied by $\{c_t\}$, then $\{\lambda v_t\}$ is implied by $\{\lambda c_t\}$. To do that we only need to verify our guess. Assume this is true for $t+j$, $j>0$. Substitute in our lifetime utility and check that it also holds for time $t$.

$$\widetilde{v}_t = \left[(1-\beta)\lambda^{\rho}c_t^{\rho} + \beta(E_t \lambda^{\alpha} v_{t+1}^{\alpha})^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \underset{\lambda \text{ is constant}}{=}$$

$$\left[(1-\beta)\lambda^{\rho}c_t^{\rho} + \lambda^{\rho}\beta(E_t \lambda^{\alpha} v_{t+1}^{\alpha})^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} = \lambda\left[(1-\beta)c_t^{\rho} + \beta(E_t \lambda^{\alpha} v_{t+1}^{\alpha})^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} =$$

$$= \lambda v_t \qquad Q.E.D.$$

To proof the next part, do the same thing,

$$\widetilde{v}_t = \left[(1-\beta)\psi^{\rho} + \beta(E_t \psi^{\alpha})^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \underset{\psi \text{ is constant}}{=} [(1-\beta)\psi^{\rho} + \psi^{\rho}\beta]^{\frac{1}{\rho}}$$

$$= \psi[(1-\beta)+\beta]^{\frac{1}{\rho}} = \psi \qquad Q.E.D.$$

2. Now we want to build a measure of the cost of fluctuations as the one described in the lecture. $\eta$ will be our measure and tells us how much do we

have to increase the consumption of the agent every period to compensate him for having variability, both because he does not like uncertainty and he does not like to substitute over time. When you have a deterministic and constant path of consumption $\{\bar{c}\}$, the ex ante lifetime utility is given by $u(\bar{c}) = (E_t\bar{c}^{\alpha})^{\frac{1}{\alpha}} = \bar{c}$ given the definition of $u()$ and the second result derived in 1.The ex ante lifetime utility of $(1+\eta)c$, given our first result, is given by $[E_t\{(1+\eta)v_t(c_t)\}^{\alpha}]^{\frac{1}{\alpha}} = (1+\eta)[E_t\{v_t(c_t)\}^{\alpha}]^{\frac{1}{\alpha}} = (1+\eta)u_0$. For the individual to be indifferent it has to be the case that

$$\begin{aligned} u(\bar{c}) &= u((1+\eta)c_t) \Leftrightarrow \\ \bar{c} &= (1+\eta)u_0 \Leftrightarrow \\ (1+\eta) &= \frac{\bar{c}}{u_0} \qquad Q.E.D. \end{aligned}$$

3. To solve this part, write our ex post lifetime utility for all periods with $\alpha = \rho$,

$$\begin{aligned} v_0 &= [(1-\beta)c_0^{\rho} + \beta(E_0 v_1^{\rho})]^{\frac{1}{\rho}} \\ v_1 &= [(1-\beta)c_1^{\rho} + \beta(E_1 v_2^{\rho})]^{\frac{1}{\rho}} \end{aligned}$$

...
Substitute $v_1$ in $v_0$ to get

$$\begin{aligned} v_0 &= \left[(1-\beta)c_0^{\rho} + \beta\{E_0\left[(1-\beta)c_1^{\rho} + \beta(E_1 v_2^{\rho})\right]^{\frac{1}{\rho}}\}^{\rho}\right]^{\frac{1}{\rho}} \\ &= \left[(1-\beta)c_0^{\rho} + \beta(1-\beta)E_0 c_1^{\rho} + \beta^2 E_0 v_2^{\rho}\right]^{\frac{1}{\rho}}, \end{aligned}$$

where I have used the law of iterated expectations to replace $E_1$ with $E_0$.If we keep replacing for $v_2, v_3, ...$we get to the general expression for a given time $t$

$$v_0 = \left[(1-\beta)\{c_0^{\rho} + \beta E_0 c_1^{\rho} + \beta^2 E_0 v_2^{\rho} + ... + \frac{\beta^t}{1-\beta}E_0 v_t^{\rho}\right]^{\frac{1}{\rho}},$$

and taking the limit when $t \to \infty$

$$v_0 = \left[(1-\beta)\left(c_0^{\rho} + E_0\sum_{t=1}^{\infty}\beta^t c_t^{\rho}\right)\right]^{\frac{1}{\rho}}.$$

given that $\beta < 1$ and that $v_t$ is finite, which is a monotonic transformation of

$$v_0 = \left( \frac{c_0^\rho}{\rho} + E_0 \sum_{t=1}^{\infty} \beta^t \frac{c_t^\rho}{\rho} \right) ...Q.E.D.$$

The intuition for this is clear. When we assume $\alpha = \rho$, we no longer have a distinction between the coefficients that capture risk aversion and intertemporal substitution and thus, we are back to the standard expected utility framework with CRRA preferences.

4. Now our consumption path is deterministic and intuitively the coefficient that captures risk aversion should not play any role, lifetime utility should not depend on risk aversion if there isn't any. That is the reason why, even without assuming $\alpha = \rho$ we get the same expression than before. Algebraically, note that the $\alpha$ cancel in our expression as we know that $v$ is deterministic. Thus we have

$$v_t = \left[ (1-\beta) c_t^\rho + \beta v_{t+1}^\rho \right]^{\frac{1}{\rho}},$$

which is the same expression we had in 3.(without the expectation term), so doing the same steps we get to the same result,

$$v_0 = \left( \frac{c_0^\rho}{\rho} + \sum_{t=1}^{\infty} \beta^t \frac{c_t^\rho}{\rho} \right).$$

Now we have to compute the lifetime utility for $c_{lh} = \{c_l, c_h, c_l, c_h, ...\}$ and $c_{hl} = \{c_h, c_l, c_h, c_l, c_h, ...\}$. I'll solve the first case, the second can be found following the same steps. From our expression we have

$$v_0(c_{lh}) = \left[ (1-\beta) \left( c_l^\rho + \beta c_h^\rho + \beta^2 c_l^\rho + ... \right) \right]^{\frac{1}{\rho}} =$$
$$\left[ (1-\beta) \left( c_l^\rho + \beta c_h^\rho \right)(1 + \beta^2 + \beta^4 + ...) \right]^{\frac{1}{\rho}} =$$
$$\left[ \frac{(1-\beta)}{1-\beta^2} (c_l^\rho + \beta c_h^\rho) \right]^{\frac{1}{\rho}} = v_{0lh}$$

and symmetrically,

$$v_{0hl} = \left[ \frac{(1-\beta)}{1-\beta^2} (c_h^\rho + \beta c_l^\rho) \right]^{\frac{1}{\rho}}.$$

As it was noticed before, the solution does not depend on $\alpha$ because the consumption path is deterministic.

But as we still have different consumption levels every period, it depends on the elasticity of substitution.

When we take the limit as $\beta \to 1$, the intuition is clear. The individual does not discount the future any more, he likes it as much as the present. So for him it does not make any difference in which order he gets the consumption levels and thus both paths of consumption give the same level of utility. Note that as we still have variability in consumption over time, the utility levels will depend on the elasticity of intertemporal substitution. Algebraically, applying L'Hopital Rule, $\lim_{\beta\to 1}\frac{1-\beta}{1-\beta^2} = \lim_{\beta\to 1}\frac{-1}{-2\beta} = \frac{1}{2}$ and then

$$v_0 = \lim_{\beta \to 1} v_{0hl} = \lim_{\beta \to 1} v_{0lh} = \left[\frac{1}{2}(c_h^\rho + c_l^\rho)\right]^{\frac{1}{\rho}} ...Q.E.D.$$

Using our equation for the welfare loss for $\overline{c} = \frac{1}{2}(c_h + c_l)$ and $u_0 = v_0$ obtained in 2 we get

$$(1+\eta) = \frac{\frac{1}{2}(c_h + c_l)}{\left[\frac{1}{2}(c_h^\rho + c_l^\rho)\right]^{\frac{1}{\rho}}}$$

Again the answer does not depend on $\alpha$. Notice that if the individual has $\infty$ elasticity of intertemporal substitution ($\rho \to 1$) then there is no loss at all. This is reasonable since it does not matter to him that the consumption varies over time. And the less willing he is to substitute over time, the bigger the loss.

If instead of a deterministic path we have uncertainty, the parameter $\alpha$ will play a role. This is the case when we assume that consumption path can take any of the two ex ante with probability $\frac{1}{2}$.

$$u_0 = [E_0\{v_0\}^\alpha]^{\frac{1}{\alpha}} =$$

$$\left[\frac{1}{2}\left(\frac{(1-\beta)}{1-\beta^2}(c_h^\rho + \beta c_l^\rho)\right)^{\frac{\alpha}{\rho}} + \frac{1}{2}\left(\frac{(1-\beta)}{1-\beta^2}(c_l^\rho + \beta c_h^\rho)\right)^{\frac{\alpha}{\rho}}\right]^{\frac{1}{\alpha}}.$$

When $\beta \to 1$ is totally indifferent between the two paths. So $u_0$ won't depend on $\alpha$ as he is not facing any uncertainty. Algebraically

$$\lim_{\beta \to 1} u_0 = \left[\frac{1}{2}\left(\frac{1}{2}(c_h^\rho + \beta c_l^\rho)\right)^{\frac{\alpha}{\rho}} + \frac{1}{2}\left(\frac{1}{2}(c_l^\rho + \beta c_h^\rho)\right)^{\frac{\alpha}{\rho}}\right]^{\frac{1}{\alpha}} =$$

$$\left[\frac{1}{2}(c_h^\rho + c_l^\rho)\right]^{\frac{1}{\rho}}.$$

5. Now we have the opposite case. Once uncertainty is resolved in time 0, we have a deterministic and constant path of consumption. Intuitively, ex ante lifetime utility should not depend on the elasticity of intertemporal substitution but should depend on risk aversion as we have ex ante uncertainty. And it shouldn't depend on the discount factor as present is the same than future. Algebraically, we now that $v_{0h}$ and $v_{0l}$ are, using the result derived in the first part of the exercise,

$$v_{0h} = c_h \text{ and } v_{0l} = c_l.$$

This implies

$$u_0 = [E_0\{v_0\}^\alpha]^{\frac{1}{\alpha}} = \left[\frac{1}{2}(c_h^\alpha + c_l^\alpha)\right]^{\frac{1}{\alpha}}$$

6. Now we have both uncertainty and variability over time and thus, lifetime utility should depend on both parameters, $\alpha$ and $\beta$. Write the expression for $u_0$,

$$\begin{aligned} u_0 &= [E_0\{v_0\}^{\alpha}]^{\frac{1}{\alpha}} = \left[\frac{1}{2}v_0(c_l)^{\alpha} + \frac{1}{2}v_0(c_h)^{\alpha}\right]^{\frac{1}{\alpha}} \\ &= \left[\frac{1}{2}\{(1-\beta)c_l^{\rho} + \beta(E_0 v_1^{\alpha})^{\frac{\rho}{\alpha}}\}^{\frac{\alpha}{\rho}} + \frac{1}{2}\{(1-\beta)c_h^{\rho} + \beta(E_0 v_1^{\alpha})^{\frac{\rho}{\alpha}}\}^{\frac{\alpha}{\rho}}\right]^{\frac{1}{\alpha}}, \end{aligned}$$

where I used the expression for $v_0$ given initially. Given that consumption is *i.i.d.* we have that

$$(E_0 v_1^{\alpha})^{\frac{1}{\alpha}} = (E_1 v_1^{\alpha})^{\frac{1}{\alpha}} = u_1$$

Because of stationary of the process and infinite horizon, the problem is the same in 0 and 1, i.e. $u_0 = u_1$.

Thus we can write

$$u_0 = \left[\frac{1}{2}\{(1-\beta)c_l^{\rho} + \beta u_0^{\rho}\}^{\frac{\alpha}{\rho}} + \frac{1}{2}\{(1-\beta)c_h^{\rho} + \beta u_0^{\rho}\}^{\frac{\alpha}{\rho}}\right]^{\frac{1}{\alpha}} \text{...}Q.E.D.$$

And as argued before, it depends on both parameters because the individual faces both uncertainty and fluctuations in consumption.

7. Solving with excel or mathlab, we obtain the following results for $\eta$:

| $\alpha \backslash \rho$ | 1 | .5 | -1 |
|---|---|---|---|
| 1 | 0.0000% | 0.0096% | 0.0384% |
| .5 | 0.0004 | 0.0100 | 0.0388% |
| -1 | 0.0016 | 0.0112% | 0.0400% |

When the agent is risk neutral ($\alpha = 1$) and has a elasticity of intertemporal substitution of infinite ($\rho = 1$), there is no loss. for a given $\rho$, the more risk avert (smaller $\alpha$), the bigger the loss. And for a given $\alpha$, the less willing to substitute, the bigger the loss. Notice that the results vary more with $\rho$, that is, the consumer is affected more by variability on the path of consumption than for uncertainty when the tolerance to any of those things is small.