



- Assets divisible at  $t = 1$ ; i.e. can sell  $(1 - f)L$  and get as a return for period  $t = 2$ ,  $fR_2$ , where  $f \in [0, 1]$ . Investor can seize assets and sell them himself if she wants to, in the event of a default by the entrepreneur
- Assets have no resale value ( $L_2 = 0$ ) at period  $t = 2$
- $(L, R_1, R_2, s)$  non-contractible. Possibly random (does not matter, everything's linear)
- Ex-Post bargaining.

## 2 Debt Contracts and Utility Possibility set for Renegotiation

A **debt contract** is a pair  $(P, T)$  where the entrepreneur (borrower or debtor  $D$ ) borrows  $B = I - w + T$  where  $T \geq 0$  can be interpreted as the "transfer" that the debtor (from now on,  $D$ ) receives **on top** of the loan  $I - w$ , and pays  $P$  at period 1 (there's no way there is some payment at  $t = 2!!$ ). As a part of the debt contract is specified that **on the event of a default of  $D$  on his debt,  $C$  is the (formal) owner of the asset.**

To get what debt contracts can be implemented under this setting of incomplete contracts, we need to understand the Bargaining game that Debtor  $D$  and the creditor  $C$  play if  $D$  decides to default, which is sometimes referred to as the **bargaining protocol**. In this paper, the bargaining game played follows the diagram in Figure 2.

## Given Realization of $(R_1, R_2, s, L)$

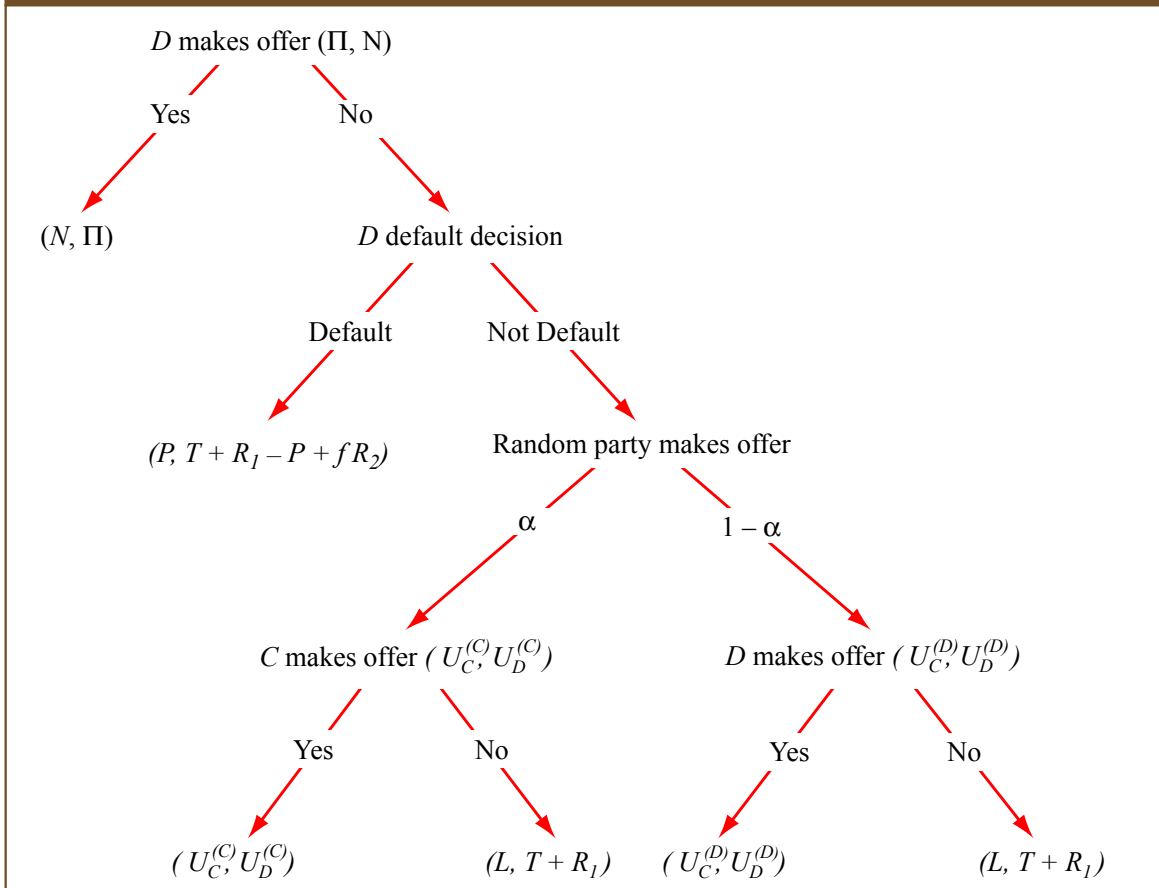


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Figure 2: Bargaining Protocol

More explicitly, the time line of the Bargaining game is as follows:

1.  $X = (R_1, R_2, L, s)$  is realized, and known by both parties
2. The debtor  $D$  "calls by phone" the creditor, offering instead of the payment of the debt ( $P$ ) payoffs of  $\Pi$  to himself and  $N$  to  $C$ .
3.  $C$  decides whether to accept the offer or not. If accepts, game ends (with no default). If not, game continues
4.  $D$  decides whether to default or to pay the debt. If he pays,  $C$  gets the value of the debt ( $P$ ), and  $D$  gets today  $T + R_1 - P$  plus what he may get tomorrow, which depends on whether he can reinvest some of the funds (if  $T + R_1 - P > 0$ ) or has to liquidate part of the asset in order to pay the debt ( $T + R_1 - P < 0$ )
5. If  $D$  decides to default, then with probability  $\alpha \in [0, 1]$   $C$  makes a "take it or leave it" offer  $(U_C^{(C)}, U_D^{(C)})$ , where  $U$  is total lifetime utility. With probability  $(1 - \alpha)$ ,  $D$  is the party making the offer

6. If the other party rejects the offer, then  $D$  defaults and  $D$  gets  $T + R_1$  while  $C$  gets  $L$  (the liquidation value of the asset).

Suppose that, in order to pay,  $D$  can liquidate assets (instead of  $C$  doing it,  $D$  does it for her)

- If  $T + R_1 + L \geq P \implies D$  can either choose to default on the debt or make payment of  $P$  (voluntary default)
- If  $T + R_1 + L < P \implies D$  defaults (involuntary default)

If  $D$  defaults (i.e. does not pay  $P$ ), what is the utility possibility frontier? Because  $s \leq \frac{R_2}{L}$ , there's a "pecking order" for the usage of funds to pay  $C$ : first with own funds ( $T + R_1$ ) and then by liquidating part of the asset.

Let  $\mathcal{U}(X) \subset \mathbb{R}_+^2$  be the utility possibility set given a realization for  $X$ . We want to trace the Pareto frontier of  $\mathcal{U}(X)$ , with generic element  $(U_1, U_2) \in \mathcal{U}(X)$

Suppose  $D$  has to give utility  $U_C$  to  $C$ , where  $U_C < T + R_1$ . Then, her maximum possible utility is

$$U_D = R_2 + s(T + R_1 - U_C) = R_2 + s(T + R_1) - sU_C$$

where  $D$  reinvests the excess funds  $T + R_1 - U_C$  at the rate  $s$

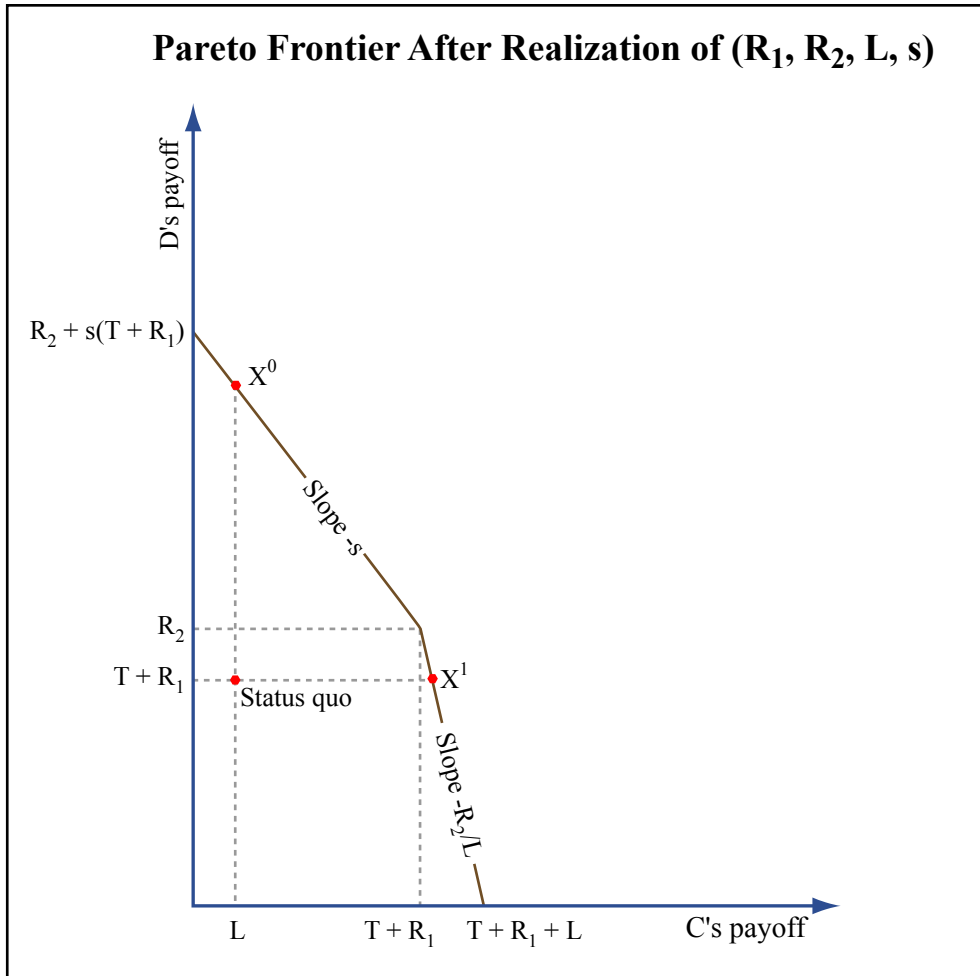
If  $U_C > T + R_1 \implies D$  has to liquidate part of the asset in order to pay  $C$ . Namely, the fraction  $f$  of the asset liquidated satisfies

$$fL = U_C - (T + R_1) \implies f = \frac{U_C - (T + R_1)}{L}$$

So, utility for  $D$  is

$$U_D = (1 - f)R_2 = (L - U_C + T + R_1) \frac{R_2}{L} \implies \frac{\partial U_D}{\partial U_C} = -\frac{R_2}{L}$$

What is the threat point? basically,  $U_C^* = L$  (gets the assets) and  $U_D^* = T + R_1$  (money in the bank)



The "kink" on the frontier occurs at the point  $(U_1, U_2) = (T + R_1, R_2)$ . At this point,  $D$  has no more extra resources to reinvest for the next period, and if  $U_1 > T + R_1$  then  $D$  has to give  $T + R_1$  entirely to  $C$ , and moreover he has to liquidate a fraction of the asset in order to pay the remaining  $U_1 - T - R_1$  to  $C$

**Two extreme cases:**

- $T + R_1 > R_2$  ( $D$  is very wealthy). If this is the case, there is never any need for liquidation, even if  $\alpha = 1$
- $T + R_1 < L$  ( $D$  is very poor). In this case, even if  $\alpha = 0$  ( $D$  has all bargaining power) there will be some liquidation.

**Note:** In FB world, there's no Pareto Frontier (we invest  $\infty$  in the technology, and we split it any way we want it)

### 3 Renegotiation Protocol and Default Decision

Suppose we have achieved the last round of bargaining (i.e.  $D$  has defaulted)

- If  $D$  is called to make an offer: she will make  $C$  indifferent between getting  $L$  or not. Looking at the graph, she offers point  $X^0$ , which gives her an utility of  $U_D = R_2 + s(T + R_1) - sL = R_2 + s(T + R_1 - L)$
- If  $C$  is called to make an offer, it depends:
  - If  $T + R_1 < R_2$  (case in the graph,  $X^1$  is below the kink) then  $C$  wants to push  $D$  into indifference between accepting the offer or getting away with  $T + R_1$  units. Because we are in this part of the graph (the one that involves liquidation) we have that

$$T + R_1 = U_D = (L - U_C + T + R_1) \frac{R_2}{L} \iff$$

$$U_C = T + R_1 + L \left( 1 - \underbrace{\frac{T + R_1}{R_2}}_f \right)$$

- If  $T + R_1 > R_2$  (case where  $X^1$  is above the kink) then no liquidation is needed.  $D$  pays  $C$  in order to maintain the asset, and the asset is not liquidated. The payoff for  $C$  is determined by equation

$$U_D = T + R_1 = R_2 + s(T + R_1) - sU_C \iff$$

$$U_C = T + R_1 - \underbrace{\frac{1}{s}(T + R_1 - R_2)}_{\text{funds } D \text{ will get after reinvesting}}$$

- Can be showed that, in general, the payoff for  $C$  is

$$U_C = \min \left\{ T + R_1 + L \left( 1 - \frac{T + R_1}{R_2} \right), T + R_1 - \frac{1}{s}(T + R_1 - R_2) \right\}$$

All these calculations were done on the subtree in which  $D$  had defaulted. Expected payoff for  $C$  given that  $D$  defaulted is:

$$\bar{P}(X, T) = \bar{P}(R_1, R_2, L, s, T) \equiv (1 - \alpha)L +$$

$$+ \alpha \min \left\{ T + R_1 + L \left( 1 - \frac{T + R_1}{R_2} \right), T + R_1 - \frac{1}{s}(T + R_1 - R_2) \right\}$$

which is the expected payoff that  $D$  has to give to  $C$ .

Then, going one step before when  $D$  makes an offer before the bargaining game starts,  $D$  would be willing to pay at most  $\bar{P}$  in order to avoid the ex-post bargaining (because Pareto Frontier has a negative slope).

**Note:** if  $\alpha = 0 \implies \bar{P} = L$  (the amount that has to be paid at  $t = 1$  by  $D$  is exactly the value of liquidation). In Kiyotaki-Moore setting, this corresponds to the situation in which  $D$  has all the bargaining power, and  $L = q_{t+1}k_t$  and  $P = Rb_t$

- The Net payoff for  $C$  is then (at the initial  $D$ 's offer)

$$N(X, P, T) = N(R_1, R_2, L, s, P, T) \equiv \min \{ \bar{P}(\cdot) - T, P - T \}$$

So that  $C$ 's gross payoff is then  $T + N$

- For  $D$ , when considering

- If  $T + N < T + R_1$ , then  $D$  pays in cash this amount, and gets  $U_D = R_2 + s(T + R_1 - (T + N)) = R_2 - s(N - R_1)$
- If  $T + N > T + R_1$ , then  $D$  has to liquidate some of the asset:  $U_D = (L - T - N + T + R_1) \frac{R_2}{L} = R_2 - \frac{(N - R_1)R_2}{L}$
- Payoff for  $D$  as a function of  $N$  is then

$$\Pi(R_1, R_2, L, s; P, T) = \min \left\{ R_2 - s(N - R_1), R_2 - \frac{(N - R_1)R_2}{L} \right\}$$

note that if  $N$  is fixed,  $\Pi$  does not depend on  $(P, T)$ !

- The liquidation function is

$$f(R_1, R_2, L, s; P, T) = \min \left\{ 1, 1 - \frac{N - R_1}{L} \right\}$$

which is the fraction of the project that  $D$  has to liquidate in the SPE of the game, as a function of  $X =$

## 4 Optimal Debt Contract

At  $t = 0$ ,  $D$  offers a contract to the investor  $C$ . Because they are competitive,  $D$  has all the bargaining power ex-ante (i.e. if makes  $C$  indifferent, she will take the contract). Note that ex-post (at  $t = 1$ ) both parties have some bargaining power at the renegotiation stage (measured by the parameter  $\alpha$ ). But this is because once the investment is done, parties  $D$  and  $C$  have an exclusive debt relationship (remember that  $C$  has formal ownership over the asset of  $D$ ). However, when  $D$  offers contracts, because there are a lot of potential investors,  $D$  gets all the ex-ante surplus.

Then, the program she must solve is

$$\max_{(P,T)} \mathbb{E}_X (\Pi (X; P, T)) \quad (OC)$$

$$s.t : \mathbb{E}_X (N (X; P, T)) \geq I - w \quad (BE)$$

Where  $(BE)$  stands for "Break Even" constraint of the investor or creditor  $C$ . Of course, in the optimal contract we must have that  $(BE)$  is binding. Next proposition shows that, for some extreme cases, all debt contracts  $(P, T)$  that satisfy  $\mathbb{E}_X (N (X; P, T)) = I - w$  are in fact, optimal

**Proposition 1 (Prop. 3 in the Paper)** *If either:*

1.  $X = (R_1, R_2, L, s)$  is non-stochastic (i.e. is known at  $t = 0$ )
2.  $L$  is non-stochastic and  $\alpha = 0$

*Then all contracts  $(P, T)$  that satisfy  $\mathbb{E}_X (N (X; P, T)) = I - w$  solve  $(OC)$*

**Proof.** Let's prove (1) first. See that if  $X$  is non-stochastic, then  $\bar{P}$  is a known constant, and so is  $N = \min \{ \bar{P} - T, P - T \}$  given  $(P, T)$ . If the contract breaks even,  $N$  is known and satisfies  $N = I - w$  (independent of the choice of  $(P, T)$ ), so the condition  $N = \min \{ \bar{P} - T, P - T \}$  can be replaced by

$$I - w = \min \{ \bar{P} - T, P - T \}$$

which is one equation in the two unknowns  $(P, T)$ . As long as some contract  $(P, T)$  exists that satisfy this constraint (for example,  $P = \bar{P} = I - w, T = 0$ ),  $\Pi$  is insensitive to  $(P, T)$ , since

$$\begin{aligned} \Pi &= \min \left\{ R_2 - s(N - R_1), R_2 - \frac{(N - R_1) R_2}{L} \right\} = \\ &\min \left\{ R_2 - s(I - w - R_1), R_2 - \frac{(I - w - R_1) R_2}{L} \right\} \end{aligned}$$



which is not a function of  $(P, T)$ , proving the desired result

For (2), note that if  $\alpha = 0 \implies \bar{P}(X) = L$ , which is assumed to be non-stochastic. Everything follows from point (1) from here. ■

#### 4.1 Application: Kiyotaki-Moore (1997)

In their setting, the debtor  $D$  is the farmer, that wants to finance an investment of capital ( $k_t = I$ ) by borrowing from the investor an amount of  $b_t$ . The (non-alienable) returns from the investment in the next period are

$$R_1 = (a + c)k_t, R_2 = 0, s = 0 \quad (1)$$

and the liquidation value of the project (i.e. selling the land in the market) is

$$L = q_{t+1}k_t \quad (2)$$

Using the previous result, any debt contract that makes the creditor break even will be optimal in this setting, so we can set  $T = 0$  and the amount  $P$  to be repaid, to make  $C$  break even is  $P = Rb_t$ . Because of non-stochasticity, we have that  $\bar{P} = P$  in the optimum, so there is no default, and

$$Rb_t = P = \bar{P} = L = q_{t+1}k_t \quad (3)$$

#### 4.2 Case with $s = 1$

Some other simple cases for which we can derive the optimal debt contracts in general are also mentioned in the paper. In this subsection, we will study the special case in which  $s = 1$  (the reinvestment technology is basically storage). In this case, we can show that the total ex-post surplus  $S = N + \Pi$  is

$$S = N + \Pi = R_1 + fR_2 + (1 - f)L = R_1 + f(R_2 - L) \quad (4)$$

Because the constraint (BE is binding in the optimum, we must have that  $\mathbb{E}(N) = I - w$ , so that maximizing  $\mathbb{E}(\Pi)$  is the same as maximizing total expected surplus, since

$$\mathbb{E}(S) = \mathbb{E}(N + \Pi) = \mathbb{E}(N) + \mathbb{E}(\Pi) \underset{\text{by (}}{=}$$

$$\begin{aligned} BEI - w + \mathbb{E}(\Pi) &= I - w + \mathbb{E}(R_1) + \mathbb{E}[f(R_2 - L)] \end{aligned}$$

since  $I - w + \mathbb{E}(R_1)$  is a constant, we can rewrite the optimal contract problem as

$$\begin{aligned} \max_{(P,T)} \mathbb{E}[f(R_2 - L)] & \quad (OC') \\ \text{s.t.} \quad \mathbb{E}(N) = I - w & \end{aligned}$$

Then, an optimal contract would be one that tries to concentrate liquidations (which may be needed to satisfy  $(BE)$ ) on those states where the social loss  $R_2 - L$  of liquidating is low.

**Proposition 2 (Prop. 2 in Paper)** *Suppose  $s \equiv 1$ . Then,*

1. *If  $R_1$  is the only stochastic variable, then an optimal contract must have  $P = \infty$  (asset is ownership of  $C$ ,  $D$  "rents" it for production)*
2. *If  $R_2$  is the only stochastic variable, then an optimal contract must have  $T = 0$  (full equity participation of  $D$ )*
3. *If  $L$  is the only stochastic variable and  $\alpha = 1$ , then an optimal contract must have  $P = \infty$*

**Intuition:** For (1), we want to prevent liquidation (reinvestment is inefficient) so as to make  $D$  better in the default states. This tends to happen when  $R_1$  is low, and this is the only source of inefficiency (because it is the only random variable). Then, because  $T$  does not affect the bargaining outcome on the effect of a default and  $P$  does not, we want to make  $T$  as big as possible, and setting  $P = \infty$  is a way of never receiving any payment from the debt contract. This is equivalent of the investor buying the project at time  $t = 0$  at a price  $T$ , and "renting" it to  $D$  at  $t = 1$ .

The intuition for part (2) is a little more complicated. If  $D$  defaults when  $R_2$  is high (i.e. when the project is efficient to carry on),  $C$  can use her bargaining power in the bargaining game to force a lot of liquidation (because she doesn't care about  $R_2$  at all, but can use the interest of  $D$  in it), since even a small fraction of the assets is worth a great deal to  $D$ . This creates more social inefficiency the higher  $R_2 - L$  is. The best way to eliminate this inefficiency is to allow  $D$  to keep  $C$  "at bay" by making a low debt payment  $P$ , so that  $D$  does not default to begin with, so they may never enter a renegotiation stage.

For part (3), the default states of  $X$  are those where  $L$  is low (because  $\bar{P} = L$  is increasing in  $L$ ). These are also the states where liquidation is very costly, since  $R_2 - L$  is high. Therefore, the slowest debt contract, which helps  $D$  in default states (who is the party with all the bargaining power) is good.

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