14.462 Lecture Notes
Aiyagari and Krusell-Smith

George-Marios Angeletos
MIT Department of Economics

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1 The Economy

- $i \in [0, 1]$.
- Employment $l(s_t) = s_t$ i.i.d. across i (but not necessarily across t), with support $\mathbf{S} = \{s_{\min}, ..., s_{\max}\}$, $s_{\min} > 0$. Let $\pi(s'|s) = \Pr(s_{t+1} = s'|s_t = s')$ and $\pi(s) = \Pr(s_t = s)$. Note that $\sum_{s'} \pi(s'|s) = 1$ for all s and $\pi(s') = \sum_s \pi(s'|s)\pi(s)$.
- Normalize $\mathbb{E}s = 1$.
- Preferences:

$$\mathbb{E}_0\mathcal{U} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$$

- Budget and borrowing constraint:

$$c_t + a_{t+1} = w_t s_t + (1 + r_t)a_t - \tau_t$$

$$
\begin{aligned}
a_t &= k_t - b_t \\
c_t &\geq 0 \\
k_t &\geq 0 \\
b_t &\leq \overline{b}_t \\
a_{t+1} &\geq -\overline{b}_t
\end{aligned}
$$

- The asset grid:

$$a_{t+1} \in \mathbf{A} = \{a^1, a^2, ..., a^N\}$$

where $a^1 = -\overline{b}$, or

$$a_{t+1} \in \mathbf{A} = [-\overline{b}, \infty).$$

- $\overline{b}$ is the borrowing limit. Either exogenous to the economy; or endogenous.

E.g.:

$$
\begin{aligned}
\overline{b}_t &= \inf_{\{s_{t+j}\}_{j=1}^{\infty}} \sum_{j=1}^{\infty} (q_{t+j}/q_t)[(w_{t+j}s_{t+j} - \tau_{t+j})] \\
&= \sum_{j=0}^{\infty} [(q_{t+j}/q_t)(w_{t+j}s_{\min} - r_{t+j}D)] \\
q_t &\equiv \frac{q_{t-1}}{1+r_t}
\end{aligned}
$$

Remark: If there is a steady state point $(w_t, r_t) \rightarrow (w, r)$, then:

$$
\begin{aligned}
\tau_t &\rightarrow \tau = rD \\
\overline{b}_t &\rightarrow \frac{ws_{\min} - rD}{r} = \frac{ws_{\min}}{r} - D
\end{aligned}
$$

2 Equilibrium

- Let

$$\Phi_t(a, s) = \Pr(a_t = a \text{ and } s_t = s)$$

denote the joint probability of a and s in period t.

- The distribution of wealth in period t is given by

$$\psi_t(a) = \sum_{s \in \mathbf{S}} \Phi_t(a, s) = \Pr(a_t = a)$$

- Market clearing:

$$K_t + D = \sum_{a \in \mathbf{A}} a\psi_t(a)$$

where D is (exogenous) government debt and K_t is aggregate (and per capita) capital.

- Equilibrium prices:

$$\begin{aligned} r_t &= f'(K_t) - \delta \equiv r(K_t) \\ &\Leftrightarrow K_t = \kappa(r_t) \end{aligned}$$

$$\begin{aligned} w_t &= f(K_t) - f'(K_t)K_t \equiv w(K_t) \\ &\Leftrightarrow w_t = \omega(r_t) \end{aligned}$$

2.1 Recursive Equilibrium

- Suppose that, in equilibrium, the law of motion for the distribution of wealth is some functional Γ s.t.:

$$\Phi_{t+1} = \Gamma(\Phi_t)$$

This means that the evolution of Φ_t is deterministic.

- Given Φ_t we can compute K_t by simply integrating:

$$K_t = \mathbf{K}(\Phi_t)$$

It follows that $w_t = w(\Phi_t)$ and $r_t = r(\Phi_t)$, as well as

$$\bar{b}_t = b(\Phi_t)$$

Then, we can expresse the problem of the household in recursive form, provided we let Φ_t be a state variable.

- A recursive equilibrium is given by (V, A, Γ) such that:

  1. V solves the Bellman equation;
     and A is the corresponding optimal choice:

$$\begin{aligned}
V(a,s,\Phi) &= \max U(c) + \beta \sum_{s' \in \mathbf{S}} V(a', s', \Phi')\pi(s'|s) \\
&\qquad s.t. \quad a' = w(\Phi')s' + [1 + r(\Phi')][a - c] - r(\Phi')D \\
&\qquad\qquad 0 \le c \le a, \quad a' \in \mathbf{A}(\Phi), \\
&\qquad\qquad \Phi' = \Gamma(\Phi) \\
A(a,s,\Phi) &= \arg\max\{...\}
\end{aligned}$$

  2. Γ is generated by A;
     that is, Γ maps Φ to Φ' such that

$$\Phi'(a', s') = \sum_{s \in \mathbf{S}} \Phi(a,s) \mathbf{1}_{[A(a,s,\Phi)=a']} \pi(s, s')$$

- The equilibrium path of the economy is then given by $\{\Phi_t\}_{t=0}^{\infty}$ such that

$$\Phi_{t+1} = \Gamma(\Phi_t),$$

  for given initial Φ_0.

- Remark: I write

$$K_{t+1} + D = \sum_{a \in \mathbf{A}} a\psi_{t+1}(a)$$

  whereas SL write

$$K_{t+1} + D = \sum_{s \in \mathbf{S}, a \in \mathbf{A}} A(a, s, \Phi_t)\Phi_t(a,s)$$

The two expressions are equivalent:

$$\begin{aligned}
K_{t+1} + D &= \sum_{a' \in \mathbf{A}} a' \psi_{t+1}(a') = \\
&= \sum_{a' \in \mathbf{A}} \sum_{s' \in \mathbf{S}} a' \Phi_{t+1}(a', s') \\
&= \sum_{a' \in \mathbf{A}} \sum_{s' \in \mathbf{S}} a' \sum_{s \in \mathbf{S}, a \in \mathbf{A}} \Phi_t(a, s) \mathbf{1}_{[A(a,s,\Phi_t)=a']} \pi(s'|s) = \\
&= \sum_{s \in \mathbf{S}, a \in \mathbf{A}} \sum_{a' \in \mathbf{A}} a' \mathbf{1}_{[A(a,s,\Phi_t)=a']} \Phi_t(a, s) \sum_{s' \in \mathbf{S}} \pi(s'|s) \\
&= \sum_{s \in \mathbf{S}, a \in \mathbf{A}} A(a, s, \Phi_t) \Phi_t(a, s)
\end{aligned}$$

2.2 Non-recursive Equilibrium

- I could alternative define an equilibrium as sequences $\{V_t, A_t\}_{t=0}^{\infty}$ and $\{K_t, R_t, w_t\}_{t=0}^{\infty}$ such that

  1. Given $\{R_t, w_t\}_{t=0}^{\infty}$, $\{V_t, A_t\}_{t=0}^{\infty}$ solve

     $$\begin{aligned}
     V_t(a, s) &= \max U(c) + \beta \sum_{s' \in \mathbf{S}} V_{t+1}(a', s') \pi(s'|s) \\
     & \qquad s.t. \quad a' = w_{t+1} s' + [1 + r_{t+1}][a - c] - r_{t+1} D \\
     & \qquad \qquad 0 \leq c \leq a, \quad a' \in \mathbf{A}(\Phi) \\
     A_t(a, s) &= \arg\max[...]
     \end{aligned}$$

     where $r_{t+1} = f'(K_{t+1})$ and $w_{t+1} = f(K_{t+1}) - f'(K_{t+1}) K_{t+1}$.

  2. $\{K_t, R_t, w_t\}_{t=0}^{\infty}$ is generated by Φ_0 and $\{A_t\}_{t=0}^{\infty}$: for all t,

     $$K_{t+1} + D = \sum_{s \in \mathbf{S}, a \in \mathbf{A}} A_t(a, s) \Phi_t(a, s),$$

     $$\Phi_{t+1}(a, s) = \sum_{s \in \mathbf{S}} \Phi_t(a, s) \mathbf{1}_{[A_t(a,s)=a']} \pi(s, s')$$

     and

     $$r_t = f'(K_t) \qquad w_t = f(K_t) - f'(K_t) K_t$$

- In my work, this approach is much easier. But not in general. Note that there is no guaranty we could write

$$K_{t+1} = G(K_t)$$

where G is stationary.

- Also, this approach proves useful in the characterization of the steady state of the economy. That's what Aiyagari does.

2.3 Steady State

- The steady-state distribution Φ is the fixed point of Γ :

$$\Phi = \Gamma(\Phi)$$

- The steady-state capital, interest rate, and wage are then computed as:

$$\begin{aligned} K &= \int a \, d\Phi(a) - D \\ r &= r(K) \\ w &= w(K) \end{aligned}$$

3 Aiyagari: Steady State

3.1 Individual Behavior

- Let the economy be at the steady state, for all t:

$$r_t = r, \quad w_t = w = \omega(r)$$

$$\overline{b}_t = \overline{b} \equiv \min\left\{b, \frac{w l_{\min}}{r} - D\right\} \equiv \overline{b}(w, r, D)$$

- Define:

$$
\begin{aligned}
x_t &\equiv a_t + \overline{b} \\
z_t &\equiv wl_t + (1+r)a_t + \overline{b} - \tau
\end{aligned}
$$

It follows that

$$z_t \equiv wl_t + (1+r)x_t - \zeta$$

where z_t are total resources available in t and x_{t+1} is investment in t and

$$\zeta \equiv r\overline{b} + \tau = r[\overline{b} + D] = \zeta(w, r, D)$$

Remark: If $\Delta\overline{b} = -\Delta D$, as in the case of the natural borrowin limit, ζ is independent of D. Otherwise, an increase in D (an increase in τ) is like a decrease in the labor income path.

- Then, for individual i:

$$
\begin{aligned}
c_t &= z_t - x_{t+1} \\
z_{t+1} &= ws_{t+1} + (1+r)x_{t+1} - \zeta
\end{aligned}
$$

Assume s_{t+1} i.i.d. across t as well.

- We can now write the value function in terms of z as:

$$
\begin{aligned}
V(z) &= \max_{0 \leq 0 \leq z} U(z - x) + \beta \sum V(z')\, \pi(s') \\
& \quad s.t. \quad z' \equiv ws' - \zeta + (1+r)x
\end{aligned}
$$

and the corresponding optimal investment as

$$
\begin{aligned}
X(z) &= \arg\max_x \{...\} \\
A(z) &= X(z) - \overline{b}
\end{aligned}
$$

Remark: If $\Delta\bar{b} = -\Delta D$, then ζ and thus $V(.)$ and $X(.)$ are independent of D, implying

$$A(z; D) = A(z; 0) + D.$$

- In general, X need not be monotonic with either w or r.

- If preferences are homothetic preference and if ζ is proportional to w, then X is proportional to w.

- Also, $X \to \infty$ as $r \to \rho$ and either $X \to -\infty$ as $r \to 0$, if no ad hoc borrowing, or $X = \bar{b}$ for all $r \leq \underline{r}$, some $\underline{r} < \rho$, if ad hoc $\bar{b}$. Thus, X is "on average" increasing.

3.2 Individual Wealth Dynamics

- We henceforth restrict to the case that s_t is i.i.d. across time and preferences are CEIS.

- Suppose for a moment that market were complete. Then, the optimal consumption rule would be given by

$$\begin{aligned} c_t &= m \cdot [(1+r)a_t + w_t s_t + h_{t+1}] = \\ &= m \cdot [z_t + (h_{t+1} - \bar{b})] \end{aligned}$$

  where h_{t+1} is the present value of labor income and m is the marginal propensity to consume out of effective wealth. Note that $m \in (0, 1)$ and $h_{t+1} >$ (natural borrowing limit) $\geq \bar{b}$. Thus

$$c_t = \bar{c} + m \cdot z_t$$

  where $\bar{c} > 0$ and $m \in (0, 1)$.

- For $z_t \leq \bar{c}/m$, $c_t > z_t$ under compelete markets, but this is impossible under incomplete markets. Under incomplete markets, $C(z)$ is bounded above by the

45^0. In particular, there is $\widehat{z} \in [z_{\min}, \overline{c}/m)$ such that $C(z) = z$ for all $z \leq \widehat{z}$ and $C(z) < z$ otherwise. Moreover, $z > \widehat{z}$, $1 > C'(z) > m$. But as $z \to \infty$, $C(z) - [\overline{c} + m \cdot z_t] \to 0$ and $C'(z) \to 0$. Finally, $C'' < 0$???

3.3 Individual Wealth Dynamics

- Given $X(.)$, the low of motion for wealth z_t of individual i is given by:

$$z_{t+1} = ws_{t+1} + (1+r)X(z_t) - \zeta$$

or

$$z' = G(z, s').$$

3.4 Steady State: General Equilibrium

- Let

$$\alpha(w, r, D) \equiv A(z; w, r, D) = E_\Phi X(z; w, r, D) - \overline{b}.$$

Remark: If $\Delta\overline{b} = -\Delta D$, then

$$\begin{aligned} \alpha(w, r, D) &= E_\Phi X(z; w, r) + D - wl_{\min}/r = \\ &= \alpha(w, r, 0) + D \end{aligned}$$

and thus $\alpha(.)$ moves one-to-one with D.

- If $\beta(1+r) \geq 1$, then $U'(c_t) \geq EU'(c_{t+1})$, which implies that $x_t, z_t, a_t \to \infty$.

Therefore, $\lim_{r \to \rho} \alpha(r) = +\infty$ and r is bounded above by $\rho \equiv 1/(1+\beta)$.

If $b = \infty$, then $\lim_{r \to 0} \overline{b}(r) = -\infty$, implying $\lim_{r \to 0} \alpha(r) = -\infty$. In that case, r is bounded below by 0.

If $b < \infty$, then $\exists\ r' > 0$ such that $\overline{b}(r) = b$ for all $r < r'$, implying that $\exists\ r'' > 0$ such that $\alpha(r) = -b$ for all $r \leq r''$ and $\alpha(r) > -b$ for all $r > r''$. In that case, $\alpha(r)$ is well defined for $r < 0$ as well.

- In equilibrium $w = \omega(r)$ and

$$a(r, D) \equiv \alpha(\omega(r), r, D)$$

That's the steady-state supply of savings, as a function of r.

- *Remark:* Even if $\alpha_r > 0$ and $\alpha_w > 0$, $\omega' < 0$, and therefore a_r is ambiguous. But we consider $a_r > 0$.

- Let

$$\kappa(r) \equiv f'^{-1}(r + \delta)$$

That's the demand for capital, as a function of r.

- *General Equilibrium:* Given D, r^* solves

$$a(r^*, D) = \kappa(r^*) + D$$

and $K^* = \kappa(r^*) \equiv f'^{-1}(r^* + \delta)$.

- *Complete vs Incomplete:*

$$\begin{aligned} r_{inco} &< 1/(1+\beta) = r_{compl} \\ &\Rightarrow K_{inco} > K_{compl} \end{aligned}$$

Saving rate $\delta K / f(K)$ also higher under incomplete markets.

- A higher $\bar{b}$ shifts $a(r)$ left and therefore K^* falls.

3.5 The Effect of Government Debt

- If $\Delta\bar{b} = -\Delta D$, then $a(r, D) = a(r, 0) + D$. In this case, r^* is determined by

$$a(r^*, 0) = \kappa(r^*)$$

and thus r^*, K^* are independent of D. (*Ricardian Equivalence*)

- If $\overline{b}$ is independent of D, then ζ increases one-to-one with $\tau = rD$. Because $-\zeta$ is like a deterministic income component, $X(.)$ raises with $-\zeta/r$ but by less than one-to-one: $\partial X(.)/\partial \zeta \approx -s/r$, where $s \in (0,1)$ is the saving rate. Therefore, an increase in D lowers $X(z)$ but by less than one-to-one: $\partial X(.)/\partial D \approx -s$. Since $a(r,D) = E_\Phi X(z;r,D) - \overline{b}$, we conclude $\partial a(r,D)/\partial D \approx -s < 0$. In this case, r^* is determined by

$$a(r^*, D) = \kappa(r^*) + D$$

and thus r^* increases with D. It follows that K^* falls with D. (*Crowding Out*)

3.6 Simulations

- Risk aversion
- Volatility of idiosyncratic shocks l
- Persistence in idiosyncratic shocks l

4 Krusell and Smith: Dynamics

- An approximate or constrained equilibrium is given by

  1. V solves the Bellman equation;
     and A is the corresponding optimal choice:

$$\begin{aligned} V(a,s,\mathbf{m}) &= \max U(c) + \beta \sum_{s' \in \mathbf{S}} V(a',s',\mathbf{m}')\pi(s'|s) \\ & \qquad s.t. \quad a' = w(\Phi)s' + [1+r(\Phi)][a-c] - r(\Phi)D \\ & \qquad\qquad c \geq a, \;\; a' \in \mathbf{A}(\Phi), \\ & \qquad\qquad \mathbf{m}' = \widehat{G}(\mathbf{m}) \\ A(a,s,\mathbf{m}) &= \arg\max\{...\} \end{aligned}$$

2. Given the initial Φ_0 and the rule A, compute $\{\mathbf{m}_t, \Phi_t\}_{t=0}^{\infty}$ by

$$\mathbf{m}_t \text{ are the moments of } \Phi_t$$

$$\Phi_{t+1}(a, s) = \sum_{s \in \mathbf{S}} \Phi_t(a, s) \mathbf{1}_{[\widehat{A}(a,s,\mathbf{m}_t)=a']} \pi(s, s').$$

The errors

$$\boldsymbol{\varepsilon}_t \equiv \mathbf{m}_{t+1} - \widehat{G}(\mathbf{m}_t)$$

are very small.

- Simulations...
- One moment (the mean) is enough...
- Wealth distribution... not enough skewness
- Introduce heterogeneity in discount factors (willingness to save)
- Discuss Rios-Rul et al.