

PROBLEM SET #1 SOLUTIONS

General Notes:

- 1) A lot of Peter's problem sets and exam questions are set up this way – basic problem in 1, then add tweaks to the model in subsequent problems. When possible, you don't have to re-derive the answers in each question.
- 2) Intuition is important. When you compare results between models, don't just compare the math. Use words.

Note on this problem set: Most of the problems came from setting up the budget constraint.

1) Everyone got this one.

$$\text{Max } \alpha \ln(C_1^i) + p^i \beta \ln(C_2^i) + \gamma \ln(1 - y^i)$$

$$\text{Subject to: } C_1^i + \left(\frac{1}{1+r}\right)C_2^i = wy^i$$

$$\frac{\alpha}{C_1^i} = \lambda$$

$$\text{FOC: } \frac{\beta p^i}{C_2^i} = \frac{\lambda}{(1+r)}$$

$$\frac{\gamma}{1-y^i} = \lambda w$$

With some algebra:

$$C_1^* = \frac{\alpha w}{\gamma + p^i \beta + \alpha}$$

$$C_2^* = \frac{p^i (1+r) \beta w}{\gamma + p^i \beta + \alpha}$$

$$y^* = \frac{\alpha + p^i \beta w}{\gamma + p^i \beta + \alpha}$$

$$\text{Lifetime utility: } \alpha \ln\left(\frac{\alpha w}{\gamma + p^i \beta + \alpha}\right) + p^i \beta \ln\left(\frac{p^i (1+r) \beta w}{\gamma + p^i \beta + \alpha}\right) + \gamma \ln\left(\frac{\gamma + p^i \beta (1-w)}{\gamma + p^i \beta + \alpha}\right)$$

2) Actuarially fair annuity.

$$\text{Max } \alpha \ln(C_1^i) + p^i \beta \ln(C_2^i) + \gamma \ln(1 - y^i)$$

$$\text{Subject to: } C_1^i + \left(\frac{p^i}{1+r}\right)C_2^i = wy^i$$

$$\frac{\alpha}{C_1^i} = \lambda$$

$$\text{FOC: } \frac{\beta p^i}{C_2^i} = \frac{\lambda p^i}{(1+r)}$$

$$\frac{\gamma}{1-y^i} = \lambda w$$

With some algebra:

$$C_1^* = \frac{\alpha w}{\gamma + p^i \beta + \alpha}$$

$$C_2^* = \frac{(1+r)\beta w}{\gamma + p^i \beta + \alpha}$$

$$y^* = \frac{\alpha + p^i \beta w}{\gamma + p^i \beta + \alpha}$$

$$\text{Lifetime utility: } \alpha \ln\left(\frac{\alpha w}{\gamma + p^i \beta + \alpha}\right) + p^i \beta \ln\left(\frac{(1+r)\beta w}{\gamma + p^i \beta + \alpha}\right) + \gamma \ln\left(\frac{\gamma + p^i \beta(1-w)}{\gamma + p^i \beta + \alpha}\right)$$

Lifetime utility has increased as long as $p^i < 1$

3) Now there are 2 types of workers, and 1 annuity to purchase. The insurance company cannot tell if you are a H type or a L type, therefore the price paid in period 1 and the benefits paid in period 2 must be the same for each type.

Assume there is a pooling equilibrium. If both buy, there will be no private savings, and all period 2 consumption will come from the annuity benefit.

Let the price = Q. Insurance company's breakeven constraint:

$$(1+r)(QC_2 + QC_2) = p^H C_2 + p^L C_2$$

$$Q = \left(\frac{p^H + p^L}{2(1+r)}\right)$$

$$\text{Max } \alpha \ln(C_1^i) + p^i \beta \ln(C_2^i) + \gamma \ln(1 - y^i)$$

$$\text{Subject to: } C_1^i + QC_2^i = wy^i$$

$$\frac{\alpha}{C_1^i} = \lambda$$

$$\text{FOC: } \frac{\beta p^i}{C_2^i} = \lambda Q = \lambda \left(\frac{p^H + p^L}{2(1+r)} \right)$$

$$\frac{\gamma}{1 - y^i} = \lambda w$$

With some algebra:

$$C_1^{i*} = \frac{\alpha w}{\gamma + p^i \beta + \alpha}$$

$$C_2^{i*} = \frac{2(1+r)\beta w}{(p^H + p^L)(\gamma + p^i \beta + \alpha)}$$

$$y^{i*} = \frac{\alpha + p^i \beta w}{\gamma + p^i \beta + \alpha}$$

Lifetime utility:

$$\alpha \ln\left(\frac{\alpha w}{\gamma + p^i \beta + \alpha}\right) + p^i \beta \ln\left(\frac{2(1+r)\beta w}{(p^H + p^L)(\gamma + p^i \beta + \alpha)}\right) + \gamma \ln\left(\frac{\gamma + p^i \beta(1-w)}{\gamma + p^i \beta + \alpha}\right)$$

When are they both better off w/ a joint annuity versus no annuity market?

$$p^i \beta \ln\left(\frac{2(1+r)\beta w}{(p^H + p^L)(\gamma + p^i \beta + \alpha)}\right) \geq p^i \beta \ln\left(\frac{(1+r)\beta w}{\gamma + p^i \beta + \alpha}\right)$$

Which holds as long as $p^H + p^L < 1$.

As long as at least 1 of the types has a survival probability < 1 , they are both better off.

Intuition: If there is an annuity market, you are pooling your mortality risk with the population. As long as 1 person dies in period 2, there will be some extra money for all the survivors, and the rate of return will be higher than saving on their own.

4) Now the government is going tax work and provide a benefit to the survivors instead of an insurance company. The benefit to this (in this model) is that the benefits in period 2 can be type-specific, and paid in proportion of their wages.

Some people opted to make the benefit as a function of their own survival probability. This was not what was intended, but it is possible. Some people also add liquidity constraints (no borrowing) in the first period. Again, possible and more realistic, but it makes the math harder, and the intuition a little murkier. I solve it assuming the benefit is a function of their average survival probabilities, and no liquidity constraints in the first period.

The government budget constraint:

$$tw(y^H + y^L)(1+r) = \frac{tw(1+r)y^H}{p} + \frac{tw(1+r)y^L}{p}$$

$$\bar{p} = \frac{p^H + p^L}{2}$$

Assume an interior solution:

$$\text{Max } \alpha \ln(C_1^i) + p^i \beta \ln(C_2^i) + \gamma \ln(1 - y^i)$$

$$\text{Subject to: } \frac{C_2^i}{1+r} \leq \frac{twy^i}{p} + (wy^i(1-t) - C_1^i)$$

$$\frac{\alpha}{C_1^i} = \lambda$$

$$\text{FOC: } \frac{\beta p^i}{C_2^i} = \frac{\lambda}{(1+r)}$$

$$\frac{\gamma}{1-y^i} = \lambda \left(\frac{tw}{p} + w(1-t) \right)$$

With some algebra:

$$C_1^{i*} = \frac{\alpha w \left(\frac{t}{p} + 1 - t \right)}{\gamma + p^i \beta + \alpha}$$

$$C_2^{i*} = \frac{p^i \beta w \left(\frac{t}{p} + 1 - t \right) (1+r)}{(\gamma + p^i \beta + \alpha)}$$

$$y^{i*} = \frac{\alpha + p^i \beta}{\gamma + p^i \beta + \alpha}$$

Lifetime utility:

$$\alpha w \left(\frac{t}{p} + 1 - t \right) + p^i \beta w \left(\frac{t}{p} + 1 - t \right) (1+r) + \gamma \ln \left(\frac{\gamma + p^i \beta (1-w)}{\gamma + p^i \beta + \alpha} \right)$$

Compare to the earlier results:

a) At what t^* does the government tax scheme yield higher utility than case 1?

$$\alpha \ln \left(\frac{t}{p} + 1 - t \right) + p^i \beta \ln \left(\frac{t}{p} + 1 - t \right) \geq 0$$

$t > 0$, and $\bar{p} < 1$

If $t=0$, get the same equilibrium as in question 1.

(no upper limit on tax rate because I allow people to borrow against their future benefits – if you assume liquidity constraints then there is an upper bound on t you must compute)

b) At what t^* does the government tax scheme yield higher utility than in case 2: personally actuarially fair annuities?

$$\alpha \ln \left(\frac{t}{p} + 1 - t \right) + p^i \beta \ln \left(p^i \left(\frac{t}{p} + 1 - t \right) \right) > 0 \text{ for } i=L,H$$

c) At what t^* does the government tax scheme yield higher utility than in case 3: pooling mortality risk with a jointly-priced annuity?

$$\alpha \ln \left(\frac{t}{p} + 1 - t \right) + p^i \beta \ln \left(p^i \left(\frac{t}{p} + 1 - t \right) \right) > p^i \beta \ln \left(\frac{1}{p} \right)$$

If the government can set the tax rate optimally, you can raise utility. In each case, can get higher utility though the financial devices available. Case 2 is only providing insurance against own mortality, case 3 pools mortality risk, and case 4 pools mortality risk and allows different consumption streams by type.