8
Proportional and derivative control

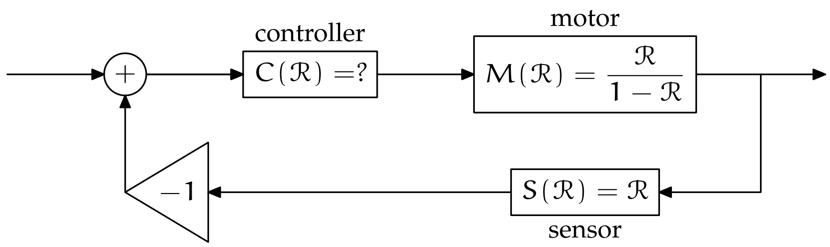
The goals of this chapter are:

- to introduce derivative control; and
- to study the combination of proportional and derivative control for taming systems with integration or inertia.

The controllers in the previous chapter had the same form: The control signal was a multiple of the error signal. This method cannot easily control an integrating system, such as the motor positioning a rod even without inertia. If the system has inertia, the limits of proportional control become even more apparent. This chapter introduces an alternative: derivative control.

8.1 Why derivative control

An alternative to proportional control is derivative control. It is motivated by the integration inherent in the motor system. We would like the feedback system to make the actual position be the desired position. In other words, it should copy the input signal to the output signal. We would even settle for a bit of delay on top of the copying. This arrangement is shown in the following block diagram:

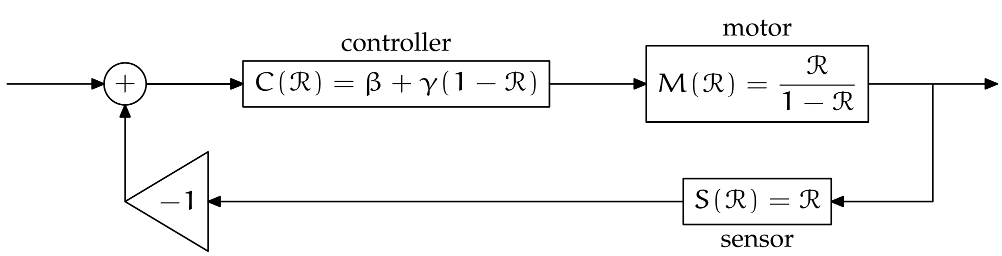


Since the motor has the functional $\mathcal{R}/(1-\mathcal{R})$, let's put a discrete-time derivative $1-\mathcal{R}$ into the controller to remove the $1-\mathcal{R}$ in the motor's denominator. With this **derivative control**, the forward-path cascade of the controller and motor contains only powers of $\mathcal{R}$. Although this method is too fragile to use alone, it is a useful idea. Pure derivative control is fragile because it uses pole–zero cancellation. This cancellation is mathematically plausible but, for the reasons explained in lecture, it produces unwanted offsets in the output. However, derivative control is still useful. As we will find, in combination with proportional control, it helps to stabilize integrating systems.

8.2 Mixing the two methods of control

Proportional control uses β as the controller. Derivative control uses $\gamma(1-\mathcal{R})$ as the controller. The linear mixture of the two methods is

$$C(\mathcal{R}) = \beta + \gamma(1-\mathcal{R}).$$



Let $F(\mathcal{R})$ be the functional for the entire feedback system. Its numerator is the forward path $C(\mathcal{R})M(\mathcal{R})$. Its denominator is $1 - L(\mathcal{R})$, where $L(\mathcal{R})$ is the loop functional or **loop gain** that results from going once around the feedback loop. Here the loop functional is

$$L(\mathcal{R}) = -C(\mathcal{R})M(\mathcal{R})S(\mathcal{R}).$$

Don't forget the contribution of the inverting (gain= -1) element! So the overall system functional is

$$F(\mathcal{R}) = \frac{(\beta + \gamma(1-\mathcal{R}))\ \frac{\mathcal{R}}{1-\mathcal{R}}}{1 + (\beta + \gamma(1-\mathcal{R}))\ \frac{\mathcal{R}}{1-\mathcal{R}}\,\mathcal{R}}.$$

Clear the fractions to get

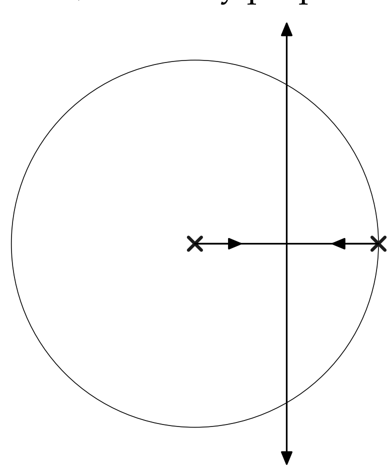
$$F(\mathcal{R}) = \frac{\text{whatever}}{1 - \mathcal{R} + (\beta + \gamma(1-\mathcal{R}))\mathcal{R}^2}.$$

The *whatever* indicates that we don't care what is in the numerator. It can contribute only zeros, whereas what we worry about are the poles. The poles arise from the denominator, so to avoid doing irrelevant algebra and to avoid cluttering up the expressions, we do not even compute the numerator as long as we know that the fractions are cleared.

The denominator is

$$1 - \mathcal{R} + (\beta + \gamma)\mathcal{R}^2 - \gamma\mathcal{R}^3.$$

This cubic polynomial produces three poles. Before studying their locations – a daunting task with a cubic – do an extreme-cases check: Take the limit $\gamma \to 0$ to turn off derivative control. The system should turn into the pure proportional-control system from the previous chapter. It does: The denominator becomes $1 - \mathcal{R} + \beta\mathcal{R}^2$, which is the denominator from Section 7.2. As the proportional gain β increases from 0 to ∞, the poles, which begin at 0 and 1, move inward; collide at $1/2$ when $\beta = 1/4$; then split upward and downward to infinity. Here is the root locus of this limiting case of $\gamma \to 0$, with only proportional control:



8.3 Optimizing the combination

We would like to make the whole system as stable as possible, in the sense that the least stable pole is as close to the origin as possible. The root locus for the general combination has three branches, one for each pole, whereas the limiting case of proportional control has only two poles and two branches. Worse, the root locus for the general combination is generated by two parameters – the gains of the proportional and the derivative portions – whereas in the limiting case it is generated by only one parameter. The general analysis seems difficult.

Surprisingly, the extra parameter rescues us from painful mathematics. To see how, look at the coefficients in the cubic:

$$1 - \mathcal{R} + (\beta + \gamma)\mathcal{R}^2 - \gamma\mathcal{R}^3.$$

The factored form is

$$(1-p_1\mathcal{R})(1-p_2\mathcal{R})(1-p_3\mathcal{R}) = 1-\underbrace{(p_1 + p_2 + p_3)}_{1}\mathcal{R}+\underbrace{(p_1p_2 + p_1p_3 + p_2p_3)}_{\beta+\gamma}\mathcal{R}^2-\underbrace{p_1p_2p_3}_{\gamma}\mathcal{R}$$

So the first constraint is

$$p_1 + p_2 + p_3 = 1,$$

showing that the center of gravity of the poles is $1/3$. That condition is independent of β and γ. So the most stable system has a triple pole at $1/3$, if that arrangement is possible. To see why that arrangement is the most stable, imagine starting from it. Now move one pole inward along the real axis to increase its stability. To preserve the invariant $p_1 + p_2 + p_3 = 1$, at least one of the other poles must move outward and become less stable. Thus it is best not to move any pole away from the triple cluster, so it is the most stable arrangement.

Exercise 42. Where does the preceding argument require that the center of gravity be independent of β and γ?

If the triple-pole arrangement is impossible, then the preceding argument, which assumed its existence, does not work. And we need lots of work to find the best arrangement of poles.

Fortunately, the triple pole is possible thanks to the extra parameter γ. Having freedom to choose β and γ, we can set the $\mathcal{R}^2$ coefficient $\beta + \gamma$ independently from the $\mathcal{R}^3$ coefficient, which is $-\gamma$. So, using β and γ as separate dials, we can make any cubic whose poles are centered on 1/3.

Let's set those dials by propagating constraints. With $p_1 = p_2 = p_3 = 1/3$, the product $p_1p_2p_3 = 1/27$. So the gain of the derivative controller is

$$\gamma = \frac{1}{27}.$$

The last constraint is that $p_1p_2 + p_1p_3 + p_2p_3 = 3/9 = 1/3$. So $\beta + \gamma = 1/3$. With $\gamma = 1/27$, this equation requires that the gain of the proportional controller be $\beta = 8/27$. The best controller is then

$$C(\mathcal{R}) = \frac{8}{27} + \frac{1}{27}(1 - \mathcal{R}) = \frac{1}{3}\left(1 - \frac{\mathcal{R}}{9}\right).$$

Exercise 43. What is the pole-zero plot of the forward path $C(\mathcal{R})M(\mathcal{R})$?

This controller has a zero at $z = 1/9$. So the added zero has pulled the poles into the sweet spot of 1/3. In comparison with pure proportional control, where the worst pole could not get closer than $z = 1/2$, derivative control has dragged the poles all the way to $z = 1/3$. A judicious amount of derivative control has helped stabilize the system.

8.4 Handling inertia

The last example showed how to use derivative control and computed how much to use. However, derivative control was not essential to stabilizing the feedback system since proportional control alone can do so and can drag the least stable pole to $z = 1/2$. But derivative control becomes essential when the system has inertia.

Without inertia, the motor accumulates angular velocity to produce angle, which is represented by the difference equation

$$y[n] = y[n-1] + x[n-1]$$

and the system functional $M(\mathcal{R}) = \mathcal{R}/(1-\mathcal{R})$. The model of inertia in Section 7.4 added a term to the motor's difference equation:

$$y[n] = y[n-1] + x[n-1] + \underbrace{m(y[n-1] - y[n-2])}_{\text{inertia}},$$

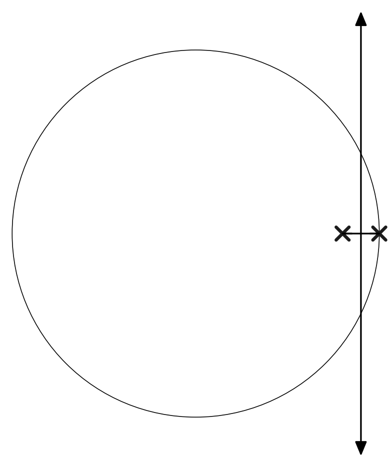
where m is a constant between 0 (no inertia) and 1 (maximum inertia). This term changes the motor's system functional to

$$M(\mathcal{R}) = \frac{1}{1-(1+m)\mathcal{R} + m\mathcal{R}^2}.$$

It factors into poles at m and 1:

$$M(\mathcal{R}) = \frac{1}{(1-m\mathcal{R})(1-\mathcal{R})}.$$

The analysis in Section 7.4 used $m = 1/2$, and then asked you to try $m = 4/5$. You should have found that the arm is hard to position when m is so close to 1. The figure shows the root locus for the motor with inertia $m = 4/5$ and controlled only using proportional control. The least stable pole can, with the right proportional gain, be dragged to the collision point $z = 0.9$. But the pole cannot be moved farther inward without moving the other pole outward. A pole at $z = 0.9$ means that the system's response contains the mode 0.9^n, which converges only slowly to zero.



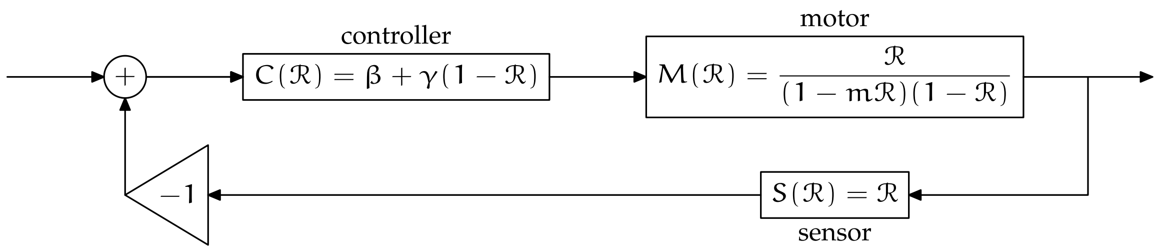
> *Pause to try 39.* How many time steps before 0.9^n has decayed roughly by a factor of e^3 (commonly used as a measure of 'has fallen very close to zero')?

The decay 0.9^n takes roughly 10 steps to fall by a factor of e. Use the greatest approximation in mathematics:

$$0.9^{10} = (1-0.1)^{10} \approx e^{-0.1\times 10} = e^{-1}.$$

So 30 time steps make the signal fall by a factor of e^3. In some applications, this wait might be too long.

Derivative control can pull the poles toward the origin, thereby hastening the convergence. Let's analyze how much derivative control to use by finding the poles of the feedback system. The feedback system is



Its system functional has the form

$$F(\mathcal{R}) = \frac{N(\mathcal{R})}{D(\mathcal{R})},$$

where the denominator is

$$\begin{aligned} D(\mathcal{R}) &= 1 - \underbrace{(-C(\mathcal{R})M(\mathcal{R})S(\mathcal{R}))}_{\text{loop functional } L(\mathcal{R})} \\ &= 1 + C(\mathcal{R})M(\mathcal{R})S(\mathcal{R}). \end{aligned}$$

In the product $C(\mathcal{R})M(\mathcal{R})S(\mathcal{R})$, the only term with a denominator is $M(\mathcal{R})$. To clear its denominator from $D(\mathcal{R})$, the whole denominator will get multiplied by the denominator of $M(\mathcal{R})$, which is $(1 - m\mathcal{R})(1 - \mathcal{R})$. So the system functional will end up with a denominator of

$$(1 - m\mathcal{R})(1 - \mathcal{R}) + \underbrace{(\beta + \gamma(1 - \mathcal{R}))}_{\text{controller}}\mathcal{R}^2.$$

After the controller come two powers of $\mathcal{R}$, one from the sensor, the other from the numerator of the motor functional $M(\mathcal{R})$. After expanding the products, the denominator is

$$1 - (1 + m)\mathcal{R} + (m + \beta + \gamma)\mathcal{R}^2 - \gamma\mathcal{R}^3.$$

This system has three parameters: the proportional gain β, the derivative gain γ, and the inertia pole m. Before spending the effort to analyze a cubic equation for its poles, check whether the equation is even reasonable! The fastest check is the extreme cases of taking parameters to zero. The limit

$m \to 0$ wipes out the inertia and should reproduce the denominator in the preceding section. In that limit, the denominator becomes

$$1 - \mathcal{R} + (\beta + \gamma)\mathcal{R}^2 - \gamma\mathcal{R}^3 \qquad (m \to 0 \text{ limit}),$$

which matches the denominator in Section 8.2. Good!

Adding the limit $\gamma \to 0$ then wipes out derivative control, which should reproduce the analysis of the simple motor with only proportional control in Section 7.2. Adding the $\gamma \to 0$ limit turns the denominator into

$$1 - \mathcal{R} + \beta\mathcal{R}^2 \qquad (m \to 0, \gamma \to 0 \text{ limit}),$$

which passes the test. Adding the $\beta \to 0$ limit wipes out the remaining feedback, leaving the bare motor functional $M(\mathcal{R})$, which indeed has a factor of $1 - \mathcal{R}$ in the denominator. So the candidate denominator passes this third test too.

Although passing three tests does not guarantee correctness, the tests increase our confidence in the algebra, perhaps enough to make it worthwhile to analyze the cubic to find where and how to place the poles. For convenience, here is the cubic again:

$$1 - (1 + m)\mathcal{R} + (m + \beta + \gamma)\mathcal{R}^2 - \gamma\mathcal{R}^3.$$

We would like to choose β and γ so that the worst pole – the one farthest from the origin – is as close as possible to the origin.

Maybe we can try the same trick (method?) that we used in the analysis without inertia: to place all three poles at the same spot. Let's assume that this solution is possible, and propagate constraints again. The sum of the poles is $1 + m$, so each pole is at $p = (1 + m)/3$. The product of the poles, p^3, is $(1 + m)^3/27$, which tells us

$$\gamma = \frac{(1 + m)^3}{27}.$$

. The sum of pairwise products of poles is $3p^2$ and is therefore $m + \beta + \gamma$. Since $3p^2$ is $(1 + m)^2/3$, the equation for β is

$$\frac{(1 + m)^2}{3} = m + \beta + \gamma.$$

So the proportional gain is:

$$\beta = \frac{(1+m)^2}{3} - m - \gamma = \frac{m^2 - m + 1}{3} - \frac{(1+m)^3}{27}.$$

To summarize,

$$\begin{aligned}\gamma &= \frac{(1+m)^3}{27},\\ \beta &= \frac{m^2 - m + 1}{3} - \frac{(1+m)^3}{27}.\end{aligned}$$

An interesting special case is maximum inertia, which is $m = 1$. Then $\gamma = 8/27$ and $\beta = 1/27$, so the controller is

$$\begin{aligned}\frac{1}{27} + \frac{8}{27}(1 - \mathcal{R}) &= \frac{1}{3} - \frac{8}{27}\mathcal{R}\\ &= \frac{1}{3}\left(1 - \frac{8}{9}\mathcal{R}\right).\end{aligned}$$

So the controller contains a zero at $8/9$, near the double pole at 1. This mixed proportional–derivative controller moves all the poles to $z = (1 + m)/3 = 2/3$, which is decently inside the unit circle. So this mixed controller can stabilize even this hard case. This case is the hardest one to control because the motor-and-rod system now contains two integrations: one because the motor turns voltage into angular velocity rather than position, and the second because of the inertia pole at 1. This system has the same loop functional as the steering-a-car example in lecture (!), which was unstable for any amount of pure proportional gain. By mixing in derivative control, all the poles can be placed at $2/3$, which means that the system is stable and settles reasonably quickly. Since

$$\left(\frac{2}{3}\right)^{2.5} \approx e^{-1},$$

the time constant for settling is about 2.5 time steps, and the system is well settled after three time constants, or about 7 time steps.

8.5 Summary

To control an integrating system, try derivative control. To control a system with inertia, also try derivative control. In either situation, do not use pure derivative control, for it is too fragile. Instead, mix proportional and derivative control to maximize the stability, which often means putting all the poles on top of each other.