Limits to Statics and Quasistatics

Reading – Haus and Melcher - Ch. 3

Outline

- Limits to Statics
- Quasistatics
- Limits to Quasistatics

Electric Fields

GAUSS

$$\oint_S \epsilon_o \overline{E} \cdot d\overline{A} = \int_V \rho dV$$
$$= Q_{enclosed}$$

FARADAY

$$\oint_C \overline{E} \cdot d\overline{l} = -\frac{d}{dt}\left(\int_S \overline{B} \cdot d\overline{A}\right)$$

Magnetic Fields

GAUSS

$$\oint_S \overline{B} \cdot d\overline{A} = 0$$

AMPERE

$$\oint_C \overline{H} \cdot d\overline{l}$$
$$= \int_S \overline{J} \cdot d\overline{A} + \frac{d}{dt} \int_S \epsilon E dA$$

For **Statics systems** both time derivatives are unimportant, and Maxwell's Equations split into decoupled electrostatic and magnetostatic equations.

Electro-quasistatic and **Magneto-quasitatic** systems arise when one (but not both) time derivative becomes important.

Quasi-static Maxwell's Equations

Electric Fields

$$\oint_S \epsilon_0 \overline{E} \cdot d\overline{A} = \int_V \rho dV$$

EQS

$$\int_C \overline{E} \cdot d\overline{l} = 0$$

Magnetic Fields

$$\oint_S \overline{B} \cdot d\overline{A} = 0$$

MQS

$$\oint_C \overline{H} \cdot d\overline{l} = \int_S \overline{J} \cdot d\overline{A} + \frac{d}{dt} \int_S \epsilon E dA$$

Coupling of Electric and Magnetic Fields

Maxwell's Equations couple H and E fields ...

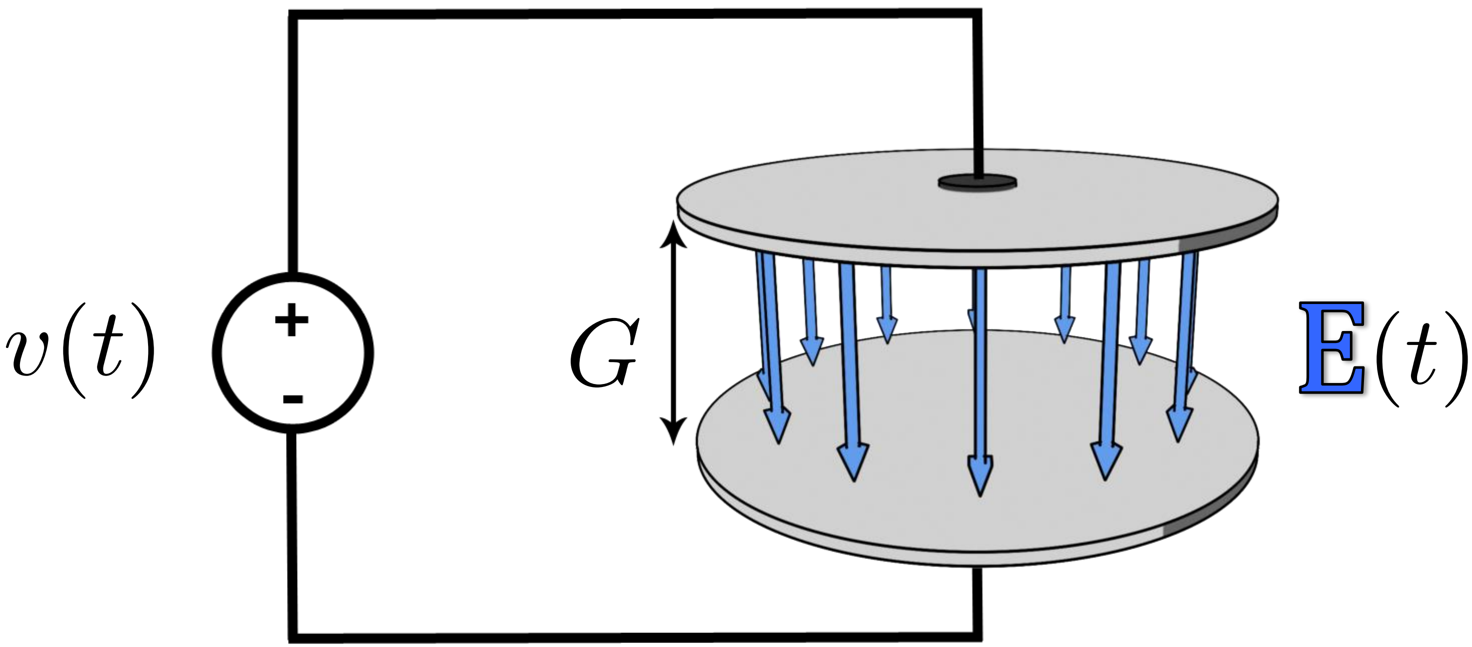
$$\oint_C \overline{E} \cdot d\overline{l} = -\frac{d}{dt}\left(\int_S \overline{B} \cdot d\overline{A}\right)$$

$$\oint_C \overline{H} \cdot d\overline{l} = \int_S \overline{J} \cdot d\overline{A} + \frac{d}{dt}\int_S \epsilon E dA$$

- When can we neglect this coupling ?
- Why do we predominantly think about E-fields for capacitors ?

  H-fields for inductors ?

"Statics" Treatment of a Capacitor



$$E(t) = v(t)/G$$

The electric field at one time depends only on the voltage at that time

Is there a magnetic field within the capacitor?

Sinusoidal Steady-State Analysis

WE WILL INTRODUCE THIS MATHEMATICAL TOOL TO HELP US ANALYZE SINUSOIDALLY CHANGING FIELDS

For the general variable $X(r,t)$

assume $X(r,t) \sim cos(\omega t)$

so that $X(r,t) = Real\left\{\tilde{X}(r)\, e^{jwt}\right\}$

Drop $Re\left\{\cdots\right\}$ and $e^{j\omega t}$ for simplicity

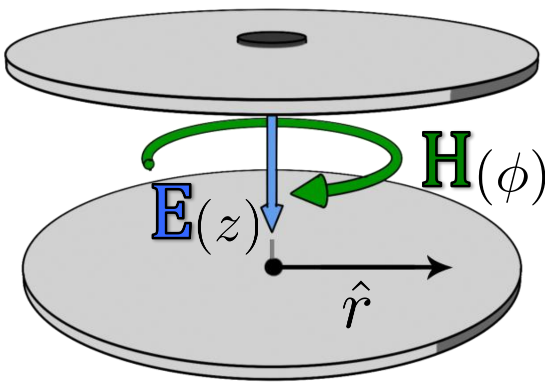
so that $X \rightarrow \tilde{X}$ and $\dfrac{\partial X(r,t)}{\partial t} \rightarrow j\omega\, \tilde{X}$

Magnetic Field inside the Capacitor

Assume SSS analysis:

$$E_z(r,t) \rightarrow \tilde{E}_o$$

$$H_\phi(r,t) \rightarrow \tilde{H}_\phi(r)$$



Ampere: $$\oint_C H \cdot dC = \int_S \frac{d\epsilon_o \overline{E}}{dt} \cdot dS$$

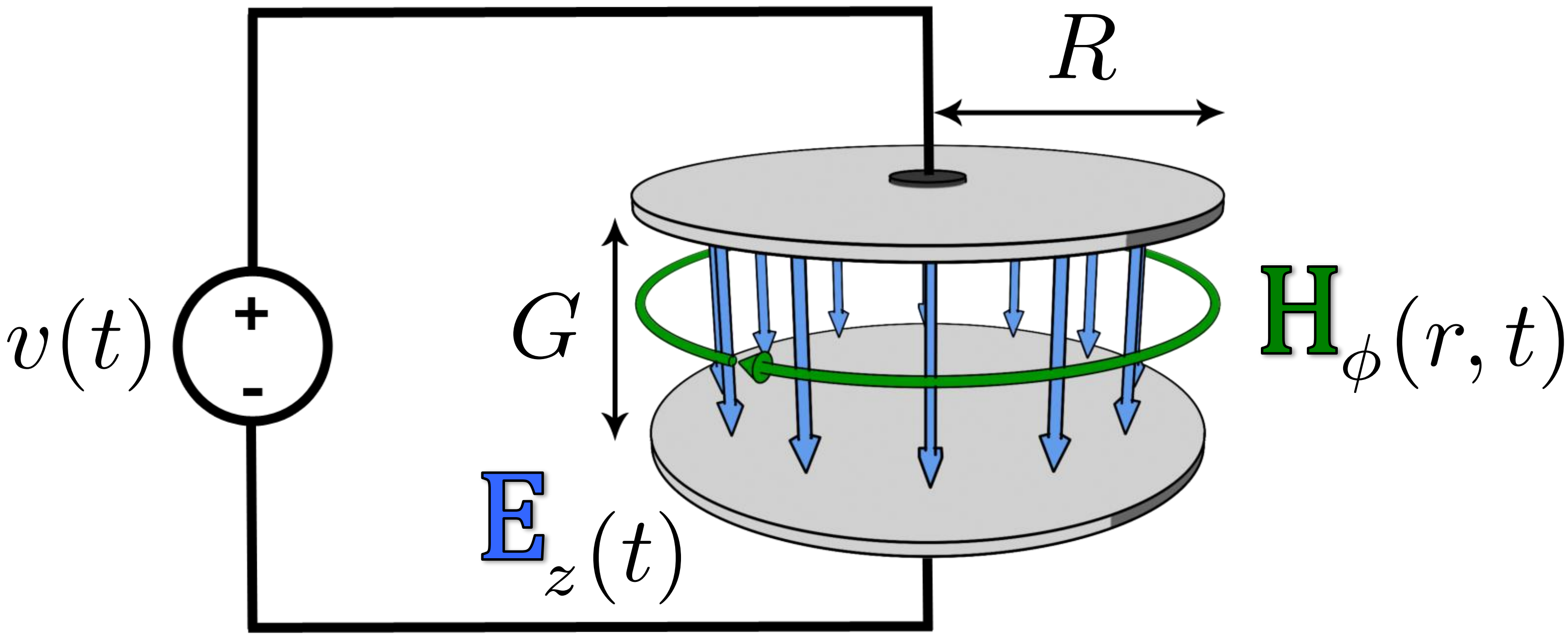
$$2\pi r \tilde{H}_\phi(r) = j\omega\epsilon_o \tilde{E}_o \pi r^2$$

$$\tilde{H}_\phi(r) = \frac{j\omega\epsilon_o r}{2}\tilde{E}_o$$

Let the contour *C* follow H_Φ and use symmetry

If the *E*-field in the capacitor is changing, it will induce the circular *H*-field

Is the electric field spatially constant within the capacitor?



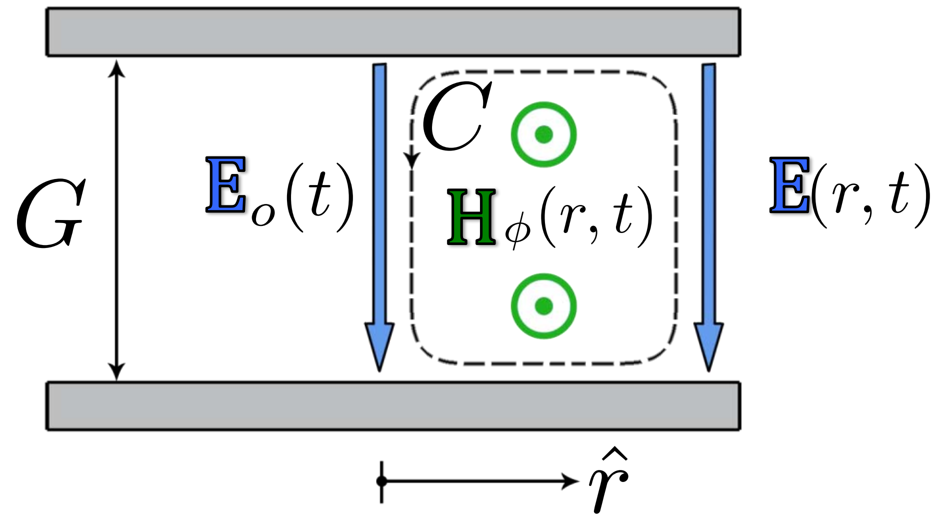
Faraday: $$\oint_C \overline{E} \cdot dC = -\int_S \frac{d\mu_o H}{dt} \cdot dS$$

The induced circular *H*-field will then induce an additional *E*-field

Corrected Electric Field inside the Capacitor

Faraday: $\oint_C \overline{E} \cdot dC$

$$= -\int_S \frac{d\mu_o H}{dt} \cdot dS$$



$$G(\tilde{E}_o - \hat{E}(r)) = -j\omega\mu_o \int_o^r \tilde{H}_\phi(r) G dS$$

$$\tilde{E}(r) = \tilde{E}_o - \frac{\epsilon_o \mu_o r^2 \omega^2}{4} \tilde{E}_o$$

The induced *E*-field will "fight" the initial *E*-field

When is the electric field correction small?

Small correction $\Rightarrow \dfrac{\epsilon_o \mu_o R^2 \omega^2}{4} \ll 1$

$$\frac{R\omega}{2c} \ll 1$$

$$T = \frac{2\pi}{\omega}$$

$$R \ll \frac{cT}{\pi}$$

$R =$ Outer radius

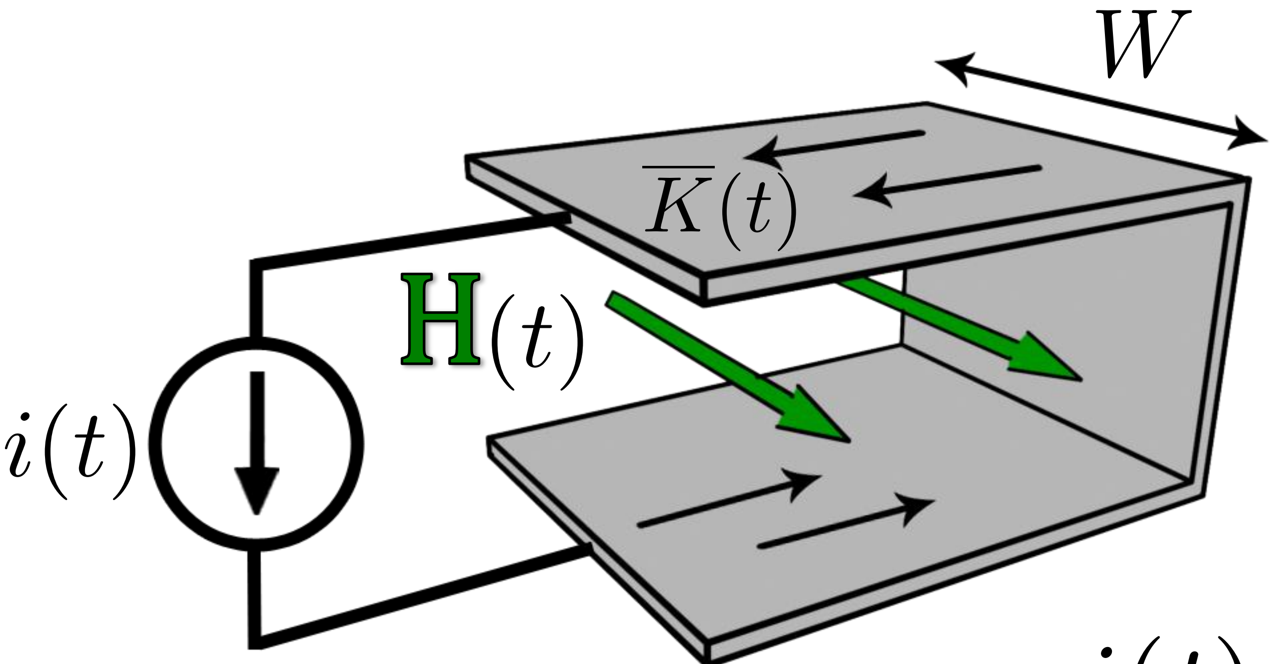
$$c = \frac{1}{\sqrt{\mu_o \epsilon_o}}$$

…Small device

$$\frac{\langle W_M \rangle}{\langle W_E \rangle} = \frac{\frac{1}{2}\mu_o \frac{1}{2}|\tilde{H}_\phi|^2}{\frac{1}{2}\epsilon_o \frac{1}{2}|\tilde{E}_\phi|^2} = \frac{\epsilon_o \mu_o r^2 \omega^2}{8} \ll 1$$

… inconsequential magnetic energy storage.

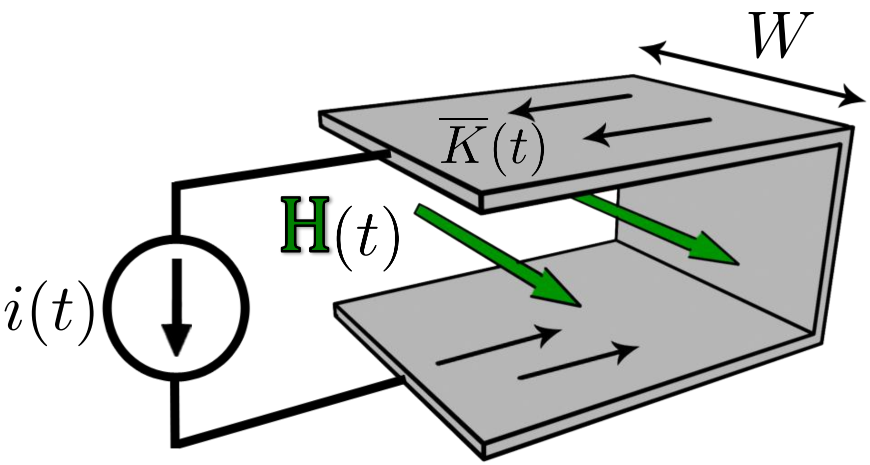
"Statics" Treatment of an Inductor

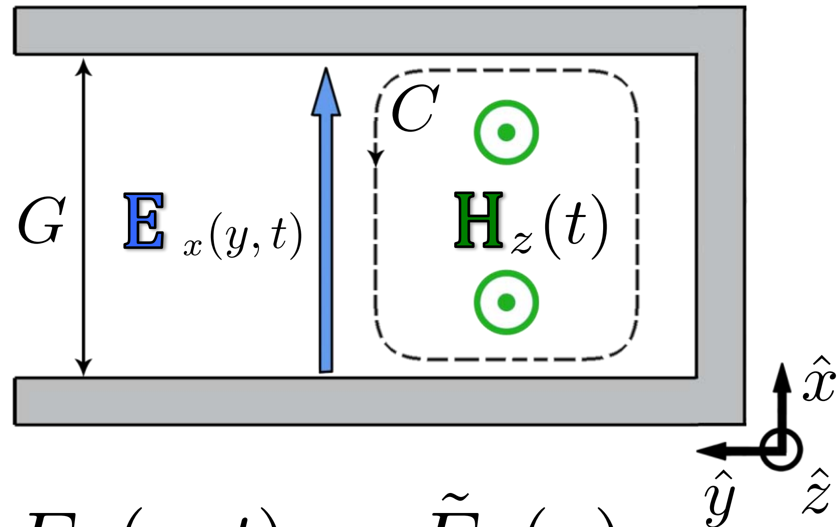


$$H(t) = K(t) = \frac{i(t)}{W}$$

The magnetic field at one time depends only on the current at that time

Electric Field Inside the Inductor





$$H_z(t) \rightarrow \tilde{H}_o \quad \& \quad E_x(y,t) \rightarrow \tilde{E}_x(y)$$

$$\oint_C \tilde{E} \cdot dC = \int_S -\frac{d\mu_o H}{dt} \cdot dS$$

$$G\tilde{E}_x(y) = j\omega\mu_o y G\tilde{H}_o$$

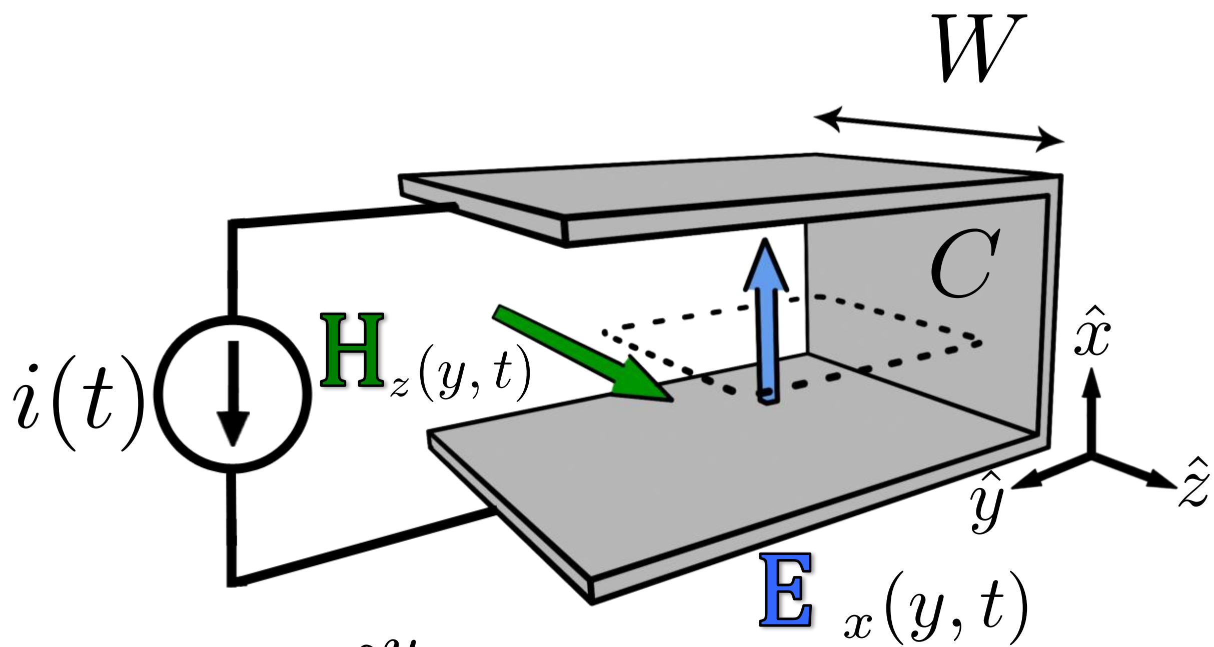
$$\tilde{E}_x(y) = j\omega\mu_o y\tilde{H}_o$$

If the *H*-field in the inductor is changing, it will induce an *E*-field

Corrected Magnetic Field inside the Inductor

Ampere:

$$\oint_C \overline{H} \cdot d\overline{C} = \int_S \frac{d\epsilon_o \overline{E}}{dt} \cdot d\overline{S}$$



$$W(\tilde{H}_z(y) - \tilde{H}_o) = j\omega \int_0^y \epsilon_o \tilde{E}_x(y) W \, dy$$

$$\tilde{H}_z(y) = \tilde{H}_o - \frac{\epsilon_o \mu_o y^2 \omega^2}{2} \tilde{H}_o$$

The induced *E*-field will then induce an additional *H*-field (that will "fight" the initial *H*-field)

When is the magnetic field correction small?

Small correction $\Rightarrow \dfrac{\epsilon_o \mu_o D^2 \omega^2}{2} \ll 1$

$$\frac{D\omega}{\sqrt{2}c} \ll 1$$

$$T = \frac{2\pi}{\omega}$$

$$D \ll \frac{cT}{\sqrt{2}\pi}$$

$D = \hat{y}$ length

$$c = \frac{1}{\sqrt{\epsilon_o \mu_o}}$$

…Small device

$$\frac{\langle W_E \rangle}{\langle W_M \rangle} = \frac{\frac{1}{2}\epsilon_o \frac{1}{2}|\tilde{E}_x(y)|^2}{\frac{1}{2}\mu_o \frac{1}{2}|\tilde{H}_o|^2} = \epsilon_o \mu_o y^2 \omega^2$$

… Inconsequential electric energy storage

Which Device is "Static"?

- Power line from Boston to Chicago operating at 60 Hz

The length of the power line is approximately 1500 km.

The wavelength of light at 60 Hz is approximately 5000 km.

- Pentium MOSFET operating at 3 GHz … ignoring any conductivity

The width of a transistor is approximately 100 nm

The wavelength of light at 3 GHz is approximately 0.1 m.

Quasistatics … One Time Derivative is Small

Electro-quasistatic (EQS)

$$\nabla \cdot \epsilon \bar{E} = \rho_F$$

$$\nabla \times \bar{E} = -\frac{\partial \mu_0 \bar{H}}{\partial t} \approx 0 \quad \text{SMALL}$$

$$\nabla \times \bar{H} = \bar{J}_F + \frac{\partial \epsilon \bar{E}}{\partial t}$$

$$\nabla \cdot \mu_0 \bar{H} = 0$$

$\Rightarrow$

$$\nabla \cdot \epsilon E = \rho_F$$

$$\nabla \times E = 0$$

$$\nabla \cdot J + \frac{\partial \rho_F}{\partial t} = 0$$

Magneto-quasitatic (MQS)

$$\nabla \cdot \mu \bar{H} = 0$$

$$\nabla \times \bar{H} = \bar{J}_F + \frac{\partial \epsilon_0 \bar{E}}{\partial t} \approx \bar{J}_F \quad \text{SMALL}$$

$$\nabla \times \bar{E} = -\frac{\partial \mu \bar{H}}{\partial t}$$

$$\nabla \cdot \epsilon_0 \bar{E} = \rho_F$$

$\Rightarrow$

$$\nabla \cdot \mu \bar{H} = 0$$

$$\nabla \times \bar{H} = \bar{J}_F$$

$$\nabla \times \bar{E} = -\frac{\partial \mu \bar{H}}{\partial t}$$

Summary: Error in Using the Quasi-static Approximation

Fields are the approximate field from the quasistatic approximation plus the induced fields that have been neglected …

$$\overline{E} = \overline{E}_{EQS} + \overline{E}_{error}$$

where

$$\oint_C \overline{E}_{error} \cdot d\overline{l} = -\frac{d}{dt}\left(\int_S \overline{B} \cdot d\overline{A}\right)$$

$$\overline{H} = \overline{H}_{EQS} + \overline{H}_{error}$$

where

$$\oint_C \overline{H}_{error} \cdot d\overline{l} = \frac{d}{dt}\int_S \epsilon E dA$$

How do we know when the errors (induced fields) are small relative to the QS fields ?

Characteristic Length and Time Scales

$$\overline{E} = \overline{E}_{EQS} + \overline{E}_{error}$$

where

$$\oint_C \overline{E}_{error} \cdot d\bar{l} = -\frac{d}{dt}\left(\int_S \overline{B} \cdot d\overline{A}\right)$$

$$\int_C \overline{H} \cdot d\bar{l}$$

$$= \int_S \overline{J} \cdot d\overline{A} + \frac{d}{dt}\int_S \epsilon E dA$$

$$HL \approx j\omega\epsilon E L^2$$

$$E_{error} L = -j\omega\mu H L^2$$

$$\frac{E_{error}}{E} = \omega^2 \mu\epsilon L^2$$

Characteristic Length and Time Scales

$$\overline{H} = \overline{H}_{EQS} + \overline{H}_{error}$$

where

$$\oint_C \overline{H}_{error} \cdot d\bar{l} = \frac{d}{dt} \int_S \epsilon E dA$$

$$\oint_C \overline{E} \cdot d\bar{l} = -\frac{d}{dt} \left(\int_S \overline{B} \cdot d\overline{A} \right)$$

$$EL \approx -j\omega\mu H L^2$$

$$H_{error} L = j\omega\epsilon E L^2$$

$$\frac{H_{error}}{H} = \omega^2 \mu \epsilon L^2$$

Error in Using the Quasi-static Approximation

$$\frac{E_{error}}{E} = \omega^2 \mu \epsilon L^2$$

$$\frac{H_{error}}{H} = \omega^2 \mu \epsilon L^2$$

For the error in using the QS approximation to be small we require …

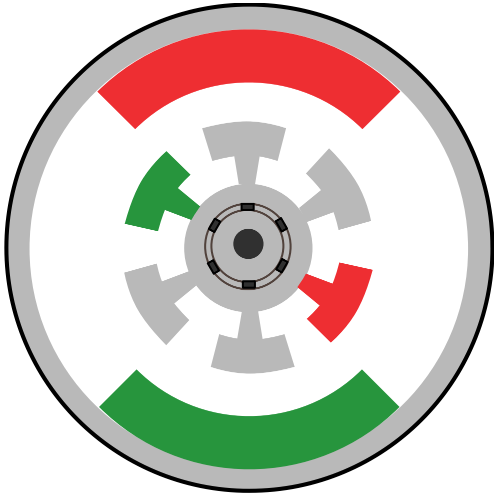
$$\omega^2 \mu \epsilon L^2 \ll 1$$

$$\omega L \ll \frac{1}{\sqrt{\mu \epsilon}}$$

EQS vs MQS for Time-Varying Fields

Why did we not worry about the magnetic field generated by the time-varying electric field of a motor ?

$$\omega L \ll \frac{1}{\sqrt{\mu\epsilon}}$$



$$\left.\begin{array}{l}\omega = 2\pi\ 2000\ \text{rpm} \\ L = 0.01\ \text{m}\end{array}\right\} \omega L = 120\ \frac{\text{m}}{\text{s}} \ll \frac{1}{\sqrt{\mu_o \epsilon_o}} \approx 3\text{x}10^8 \frac{\text{m}}{\text{s}}$$

for free-space

As another example, note:

At 60 Hz, the wavelength (typical length) in air is 5000 km, therefore, almost all physical 60-Hz systems in air are quasistatic (since they are typically smaller than 5000 km in size)

KEY TAKEAWAYS

Maxwell's Equations couple H and E fields ...

$$\oint_C \overline{E} \cdot d\bar{l} = -\frac{d}{dt}\left(\int_S \overline{B} \cdot d\overline{A}\right)$$

$$\oint_C \overline{H} \cdot d\bar{l} = \int_S \overline{J} \cdot d\overline{A} + \frac{d}{dt}\int_S \epsilon E dA$$

For the error in using the Quasi-Static approximation to be small we require ...

$$\omega^2 \mu \epsilon L^2 \ll 1$$

$$\omega L \ll \frac{1}{\sqrt{\mu\epsilon}}$$

for free-space $\frac{1}{\sqrt{\mu_o \epsilon_o}} \approx 3\text{x}10^8 \frac{\text{m}}{\text{s}}$

EQS Limits

Approach:
J = σ E ; Del ~ 1/δ ;
∂/∂t ~ 1/T ; require small electric field correction

Ampere $\Rightarrow \dfrac{H}{\delta} \sim \sigma E + \dfrac{\epsilon E}{T}$

Faraday $\Rightarrow \dfrac{E_{correction}}{\delta} \sim \dfrac{\mu_o H}{T} \sim \dfrac{\mu_o \sigma E \delta}{T} + \dfrac{\epsilon \mu_o E \delta}{T^2}$

Small Correction $\Rightarrow E >> \dfrac{\mu_o \sigma \delta^2 E}{T}$ and $\dfrac{\epsilon \mu_o \delta^2 E}{T^2}$

$T \gg \mu_o \sigma \delta^2$ … very fast magnetic diffusion

$T \gg \dfrac{\delta}{c}$ … very fast wave propagation

MQS Limits

Approach:
$J = \sigma E$; Del ~ $1/\delta$; $\partial/\partial t$ ~ $1/T$; require small magnetic field correction

Faraday $\Rightarrow \dfrac{E}{\delta} \backsim \dfrac{\mu H}{T}$

Ampere $\Rightarrow \dfrac{H_{correction}}{\delta} \backsim \dfrac{\epsilon_o E}{T} \backsim \dfrac{\epsilon_o \mu \delta H}{T^2}$

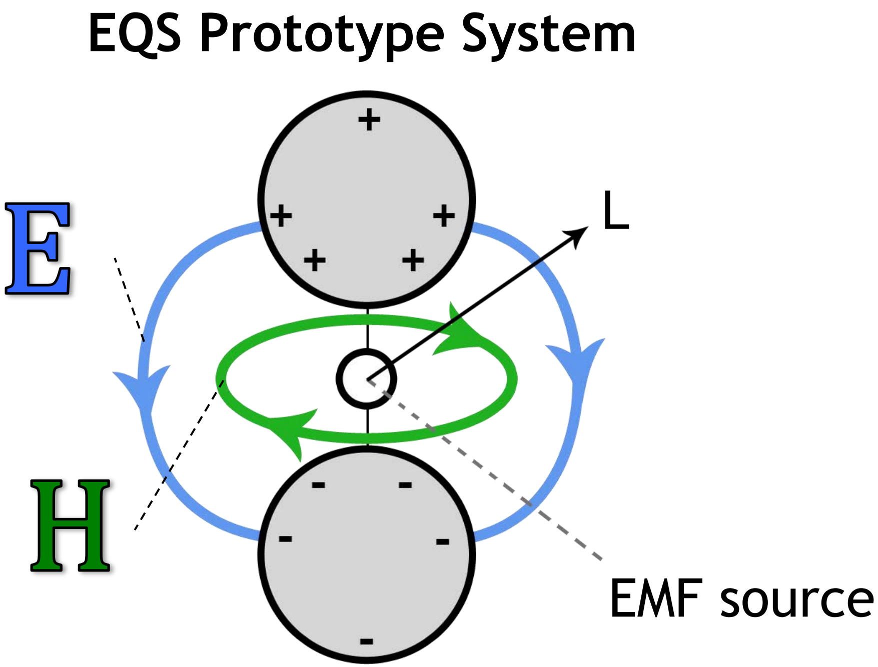
Small correction $\Rightarrow \dfrac{\epsilon_o \mu \delta H}{T^2} \ll H$ and σE ; $\sigma E \backsim \dfrac{\mu \sigma \delta H}{T}$

$T \gg \dfrac{\epsilon_o}{\sigma}$ … very fast charge relaxation

$T \gg \dfrac{\delta}{c}$ … very fast wave propagation

Satisfied in small devices with high conductivity

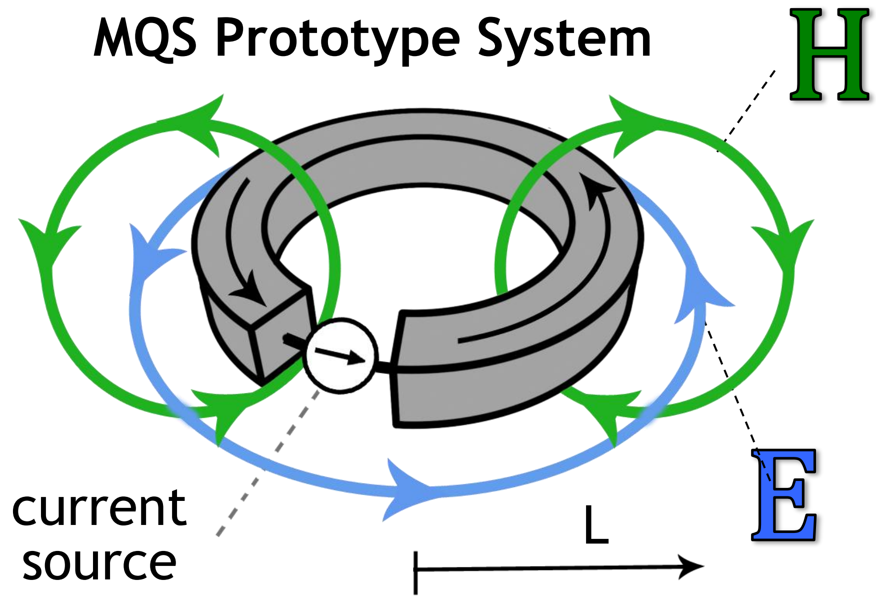
Characteristic Length and Time Scales



Source of EMF drives a pair of perfectly conducting spheres having radius and spacing on the order of ***L***

$$\int_S \epsilon_o \overline{E} \cdot d\overline{A} = \int_V \rho dV$$

$$\epsilon_o E L^2 \approx \rho L^3$$



Current source drives perfectly conducting loop with radius of the loop and cross-section on the order of ***L***

$$\int_C \overline{H} \cdot d\bar{l} \approx \int_S \overline{J} \cdot d\overline{A}$$

$$HL \approx JL^2$$

6.007 Electromagnetic Energy: From Motors to Lasers
Spring 2011