State-Space Models, Equilibrium, Linearization

6.011, Spring 2018

Lec 5

State variables are (relevant) "memory" variables

In physical systems, the natural state variables are typically related to energy storage mechanisms:

capacitor voltages or charges,
inductor currents or fluxes,
positions and velocities of masses,
…

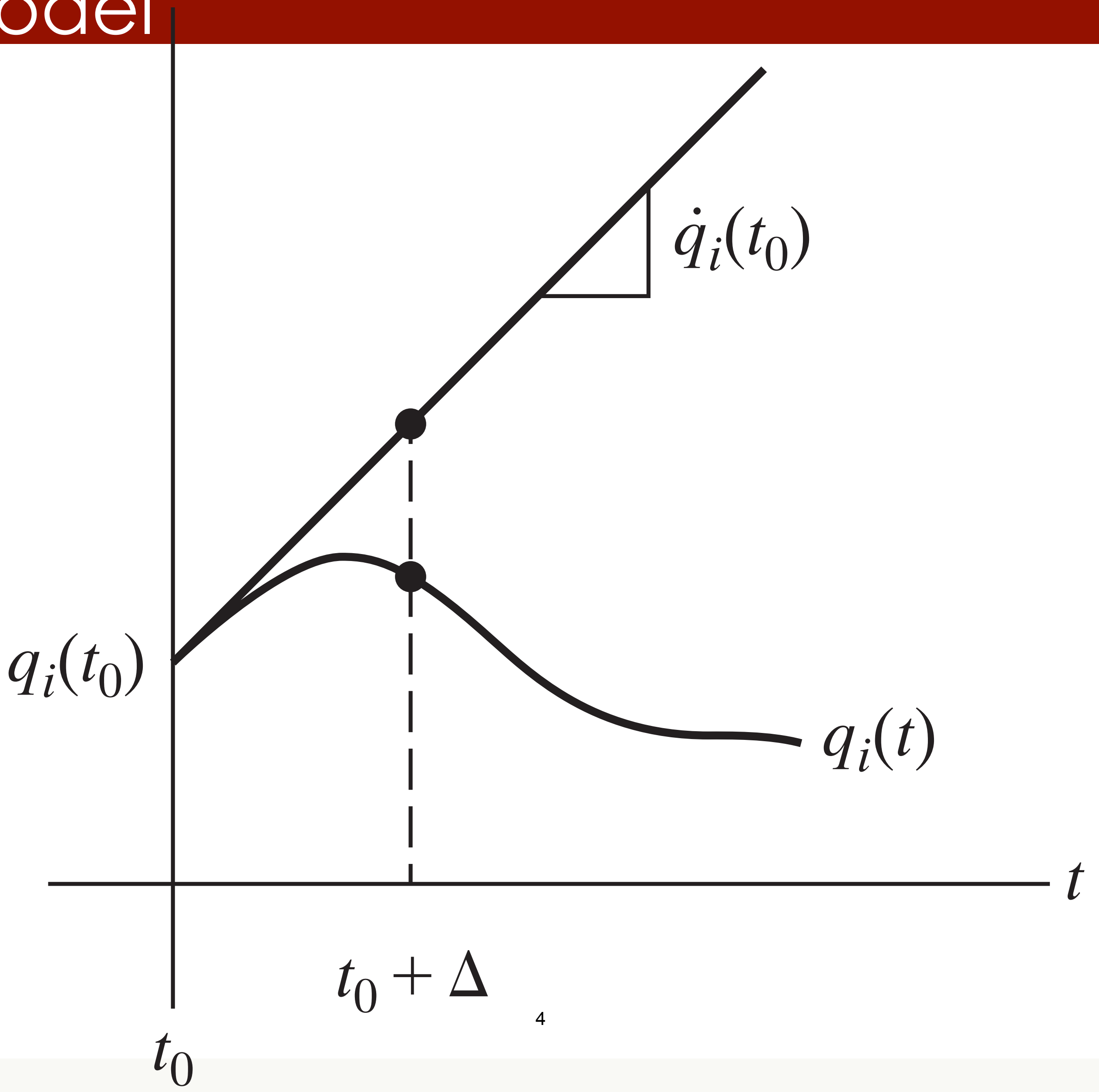
Defining properties of CT state-space models

$$\dot{\mathbf{q}}(t) = \mathbf{f}\Big(\mathbf{q}(t), x(t), t\Big)$$

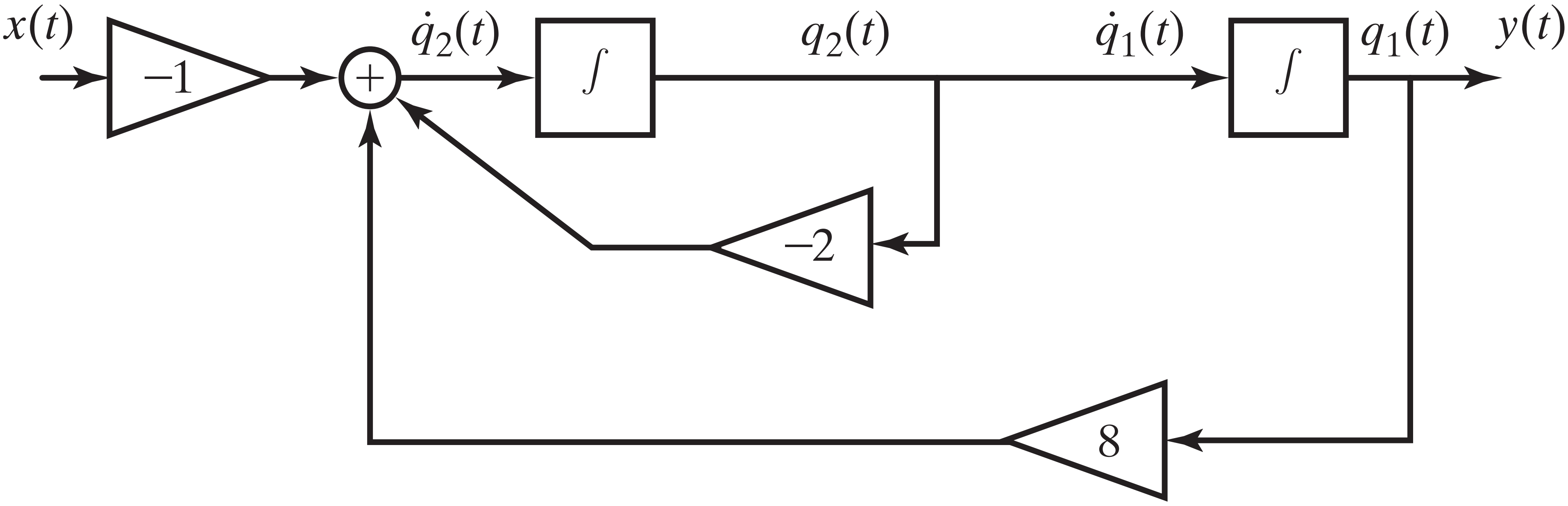
$$y(t) = g\Big(\mathbf{q}(t), x(t), t\Big)$$

- **State evolution property**
- **Instantaneous output property**

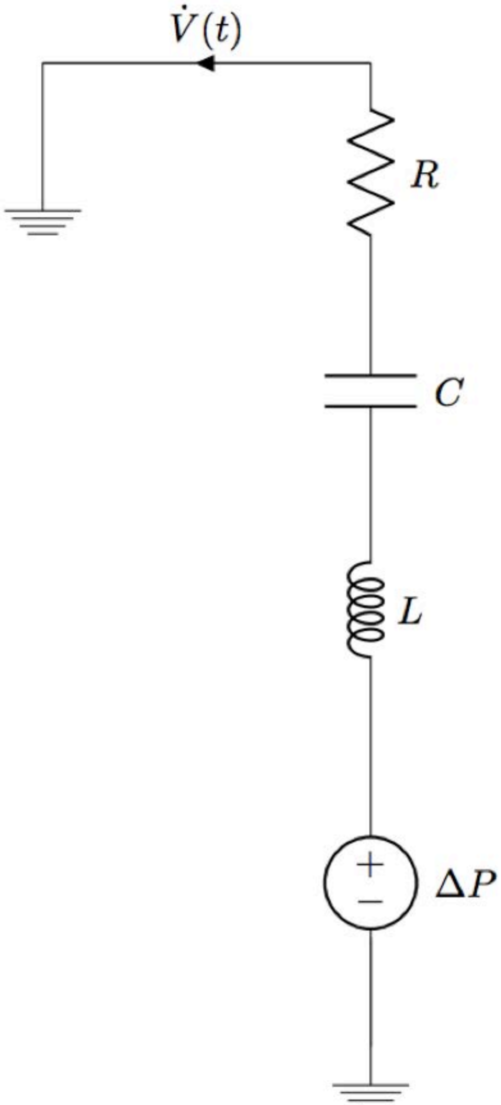
Numerical solution of CT state-space model

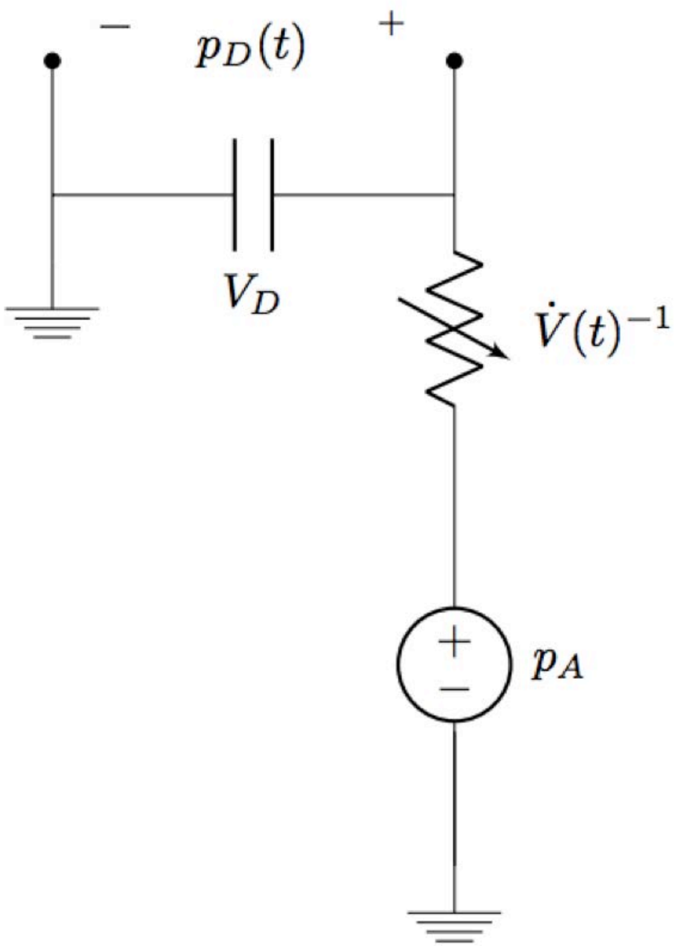


Integrator-adder-gain system



Mechanistic model for capnography





… and the governing equations

$$L\ddot{V}(t) + R\dot{V}(t) + \frac{V(t)}{C} = \Delta P$$

$$\dot{p}_D(t) = \frac{-p_D(t) + p_A}{V_D}\,\dot{V}(t)\,, \quad \dot{V}(t) > 0$$

$$\dot{p}_D(t) = \frac{p_D(t)}{V_D}\quad \dot{V}(t)\,, \quad \dot{V}(t) < 0$$

Equilibrium

For a time-invariant nonlinear system with a constant input, an initial state that the system remains at:

$$\mathrm{DT}: \quad \bar{\mathbf{q}} = \mathbf{f}(\bar{\mathbf{q}}, \bar{x})$$

$$\mathrm{CT}: \quad \mathbf{0} = \mathbf{f}(\bar{\mathbf{q}}, \bar{x})$$

Linearization at an equilibrium yields an LTI model

$$\text{DT case:} \quad \mathbf{q}[n] = \bar{\mathbf{q}} + \widetilde{\mathbf{q}}[n] \ , \quad x[n] = \bar{x} + \widetilde{x}[n] \ ,$$

$$\mathbf{q}[n+1] = \mathbf{f}(\mathbf{q}[n], x[n])$$

$$\downarrow$$

$$\widetilde{\mathbf{q}}[n+1] \approx \left[\frac{\partial \mathbf{f}}{\partial \mathbf{q}} \Big|_{\bar{\mathbf{q}}, \bar{x}} \right] \widetilde{\mathbf{q}}[n] + \left[\frac{\partial \mathbf{f}}{\partial x} \Big|_{\bar{\mathbf{q}}, \bar{x}} \right] \widetilde{x}[n]$$

for small perturbations $\widetilde{\mathbf{q}}[n]$ and $\widetilde{x}[n]$ from equilibrium

Linearization at an equilibrium yields an LTI model

$$\text{CT case:} \quad \mathbf{q}(t) = \bar{\mathbf{q}} + \widetilde{\mathbf{q}}(t)\ , \quad x(t) = \bar{x} + \widetilde{x}(t)\ ,$$

$$\dot{\mathbf{q}}(t) = \mathbf{f}(\mathbf{q}(t), x(t))$$

$$\downarrow$$

$$\dot{\widetilde{\mathbf{q}}}(t) \approx \left[\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\Big|_{\bar{\mathbf{q}},\bar{x}}\right] \widetilde{\mathbf{q}}(t) + \left[\frac{\partial \mathbf{f}}{\partial x}\Big|_{\bar{\mathbf{q}},\bar{x}}\right] \widetilde{x}(t)$$

for small perturbations $\widetilde{\mathbf{q}}(t)$ and $\widetilde{x}(t)$ from equilibrium