Matrix exponential, ZIR+ZSR, transfer function, hidden modes, reaching target states

6.011, Spring 2018

Lec 8

Modal solution of driven DT system

$$\mathbf{q}[n+1] = \mathbf{V}\boldsymbol{\Lambda}\underbrace{\mathbf{V}^{-1}\mathbf{q}[n]}_{\mathbf{r}[n]} + \mathbf{b}x[n]\,, \qquad y[n] = \mathbf{c}^T\mathbf{q}[n] + \mathtt{d}x[n]$$

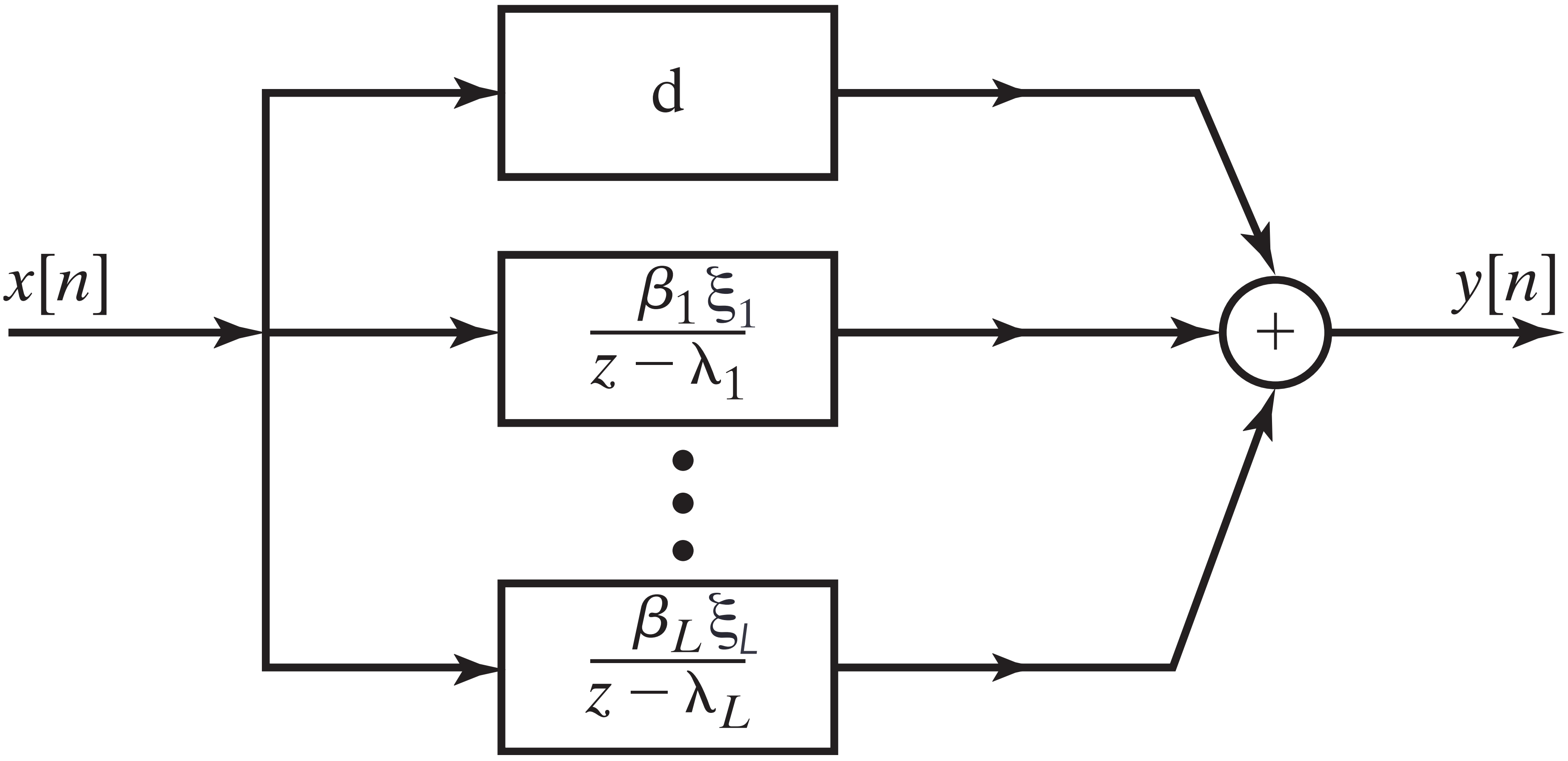
$$\downarrow$$

$$\mathbf{r}[n+1] = \boldsymbol{\Lambda}\mathbf{r}[n] + \underbrace{\mathbf{V}^{-1}\mathbf{b}}_{\beta}\,x[n]\,, \qquad y[n] = \underbrace{\mathbf{c}^T\mathbf{V}}_{\xi^T}\mathbf{r}[n] + \mathtt{d}x[n]$$

Because $\boldsymbol{\Lambda}$ is diagonal, we get the decoupled scalar equations

$$r_i[n+1] = \lambda_i r_i[n] + \beta_i x[n]\,, \qquad y[n] = \left(\sum_1^L \xi_i r_i[n]\right) + \mathtt{d}[n]$$

Underlying structure of LTI DT state-space system with L distinct modes



Reachability and Observability

$$r_i[n+1] = \lambda_i r_i[n] + \beta_i x[n] \ , \qquad y[n] = \Big(\sum_1^L \xi_i r_i[n]\Big) + \mathtt{d}[n]$$

for $i = 1, 2, \ldots, L$

$\downarrow$

$\beta_j = 0$, the $j\mathtt{th}$ mode cannot be excited from the input
i.e., the $j\mathtt{th}$ mode is **unreachable**

$\xi_k = 0$, the $k\mathtt{th}$ mode cannot be seen in the output
i.e., the $k\mathtt{th}$ mode is **unobservable**

Hidden modes

$$H(z) = \left(\sum_{i=1}^{L} \frac{\beta_i \xi_i}{z - \lambda_i} \right) + \mathtt{d}$$

Any modes that are unreachable ($\beta_i = 0$)

or/and unobservable ($\xi_i = 0$)

are "hidden" from the input-output transfer function.

ZIR + ZSR

$$r_i[n] = \lambda_i r_i[n-1] + \beta_i x[n-1]$$

$$\downarrow$$

$$r_i[n] = \underbrace{(\lambda_i^n) r_i[0]}_{ZIR} + \underbrace{\sum_{k=1}^{n} \lambda_i^{k-1} \beta_i \, x[n-k]}_{ZSR}, \quad n \geq 1$$

$$\downarrow$$

$$\mathbf{q}[n] = \sum_{i=1}^{L} \mathbf{v}_i \, r_i[n]$$

More directly ...

$$\mathbf{q}[n] = \mathbf{A}\mathbf{q}[n-1] + \mathbf{b}x[n-1]$$

$$\downarrow$$

$$\mathbf{q}[n] = \underbrace{(\mathbf{A}^n)\,\mathbf{q}[0]}_{ZIR} + \underbrace{\sum_{k=1}^{n} \mathbf{A}^{k-1}\mathbf{b}\,x[n \quad k]}_{ZSR}\,, \quad n \quad 1$$

(linear **jointly** in initial state **and** input sequence)

Similarly for CT systems

$$\dot{r}_i(t) = \lambda_i r_i(t) + \beta_i x(t)$$

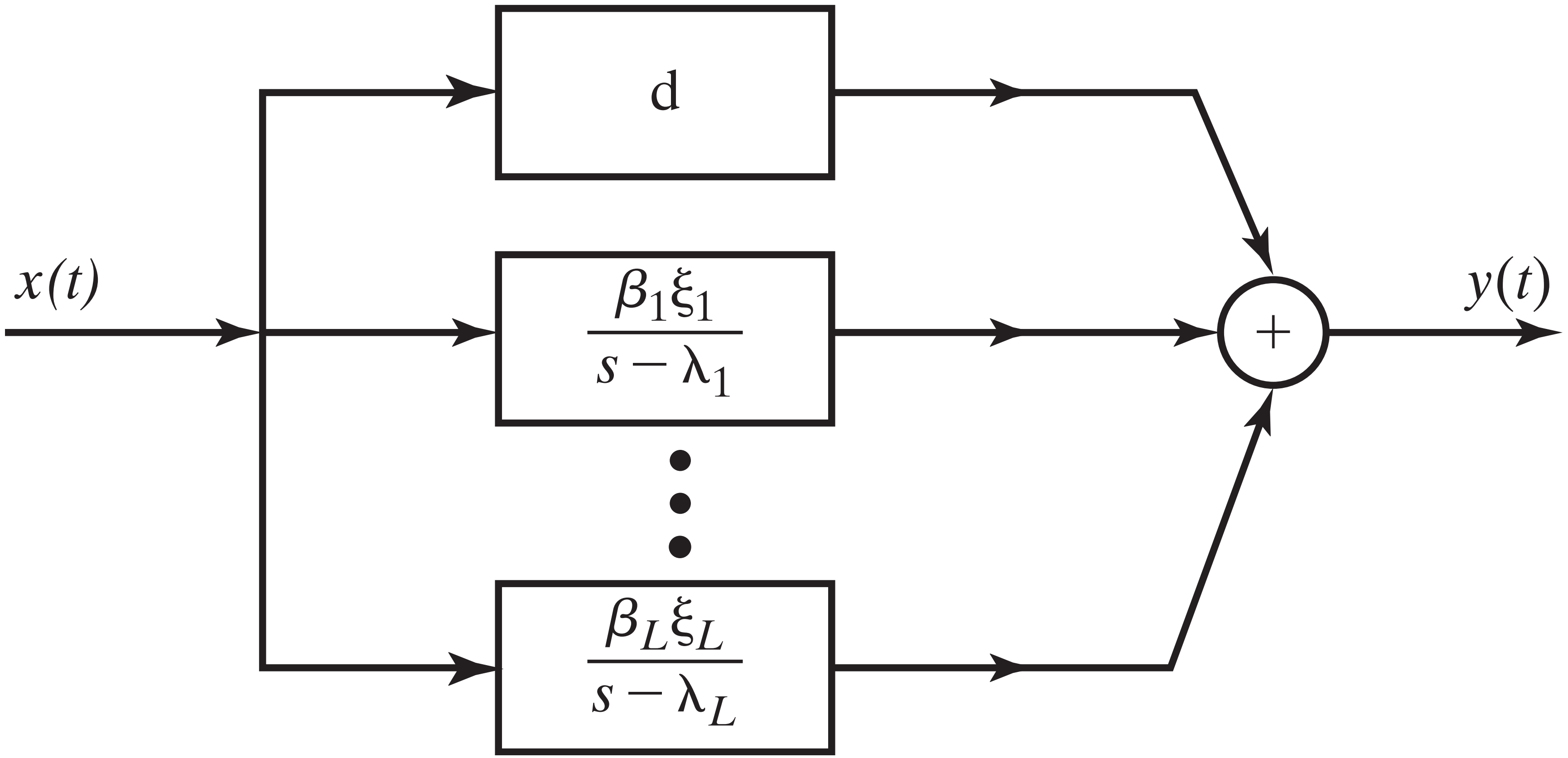
$$\downarrow$$

$$r_i(t) = \underbrace{(e^{\lambda_i t}) r_i(0)}_{ZIR} + \underbrace{\int_0^t e^{\lambda_i \tau} \beta_i \, x(t-\tau) \, d\tau}_{ZSR} , \quad t \geq 0$$

$$\downarrow$$

$$\mathbf{q}(t) = \sum_{i=1}^{L} \mathbf{v}_i \, r_i(t)$$

Decoupled structure of CT LTI system in modal coordinates



More generally

$$\mathbf{q}(t) = \underbrace{\left(e^{\mathbf{A}t}\right)\mathbf{q}(0)}_{ZIR} + \underbrace{\int_0^t e^{\mathbf{A}\tau}\mathbf{b}\, x(t \quad \tau)\, d\tau}_{ZSR}, \quad t \quad 0$$

where

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \cdots$$

$$= \mathbf{V}\, e^{\mathbf{\Lambda}t}\, \mathbf{V}^{-1}$$

Key properties of matrix exponential

$$e^{\mathbf{A}.0} = \mathbf{I}$$

$$\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t} = e^{\mathbf{A}t} \mathbf{A}$$

$$e^{\mathbf{A}t_1} e^{\mathbf{A}t_2} = e^{\mathbf{A}(t_1 + t_2)}$$

$$\text{but} \quad e^{\mathbf{A}_1} e^{\mathbf{A}_2} \neq e^{\mathbf{A}_1 + \mathbf{A}_2}$$

unless the two matrices commute

In the transform domain …

The matrix extension of

$$e^{at} \leftrightarrow \frac{1}{s - a}$$

is

$$e^{\mathbf{A}t} \leftrightarrow (s\mathbf{I} - \mathbf{A})^{-1}$$

Input-output transfer function:

$$H(s) = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{d}$$

Reaching a target state from the origin (e.g., in a 2nd-order system)

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \ , \ \mathbf{q}[0] = \mathbf{0}$$

$$\mathbf{b} = \mathbf{v}_1\beta_1 + \mathbf{v}_2\beta_2$$

Reaching a target state in 2 steps:

$$\mathbf{q}[2] = \mathbf{v}_1\gamma_1 + \mathbf{v}_2\gamma_2$$

$$\Downarrow$$

$$\begin{aligned} \begin{bmatrix} x[1] \\ x[0] \end{bmatrix} &= \begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \\ &= \frac{1}{\quad} \begin{bmatrix} \lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1/\beta_1 \\ \gamma_2/\beta_2 \end{bmatrix} . \end{aligned}$$