6.013/ESD.013J — Electromagnetics and Applications	Fall 2005
Problem Set 3 - Solutions	
Prof. Markus Zahn	MIT OpenCourseWare

Problem 3.1

A

The idea here is similar to applying the chain rule in a 1D problem:

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \left[\frac{d}{df}\left(\frac{1}{f(x)}\right)\right]\left[\frac{df}{dx}\right] = -\frac{f'(x)}{f^2(x)},$$

where $f(x)$ corresponds to $|\mathbf{r} - \mathbf{r}'|$.

So, by differentiating $f(x)$ we get part of the answer to the derivative of $1/f(x)$. But, we can just do it directly:

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\boldsymbol{\nabla}\left[\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right] = \hat{\mathbf{e}}_x \frac{\partial}{\partial x}\left[\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right] + \hat{\mathbf{e}}_y \frac{\partial}{\partial y}\left[\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right] + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\left[\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right]$$

So, we can apply the trick above by just considering x, y, and z components separately.

$$\begin{aligned}\frac{\partial}{\partial x}|\mathbf{r} - \mathbf{r}'| &= \frac{\partial}{\partial x}\left(\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right)\\ &= \frac{x-x'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\\ &= \frac{x-x'}{|\mathbf{r} - \mathbf{r}'|}\end{aligned}$$

Similarly:

$$\begin{aligned}\frac{\partial}{\partial y}|\mathbf{r} - \mathbf{r}'| &= \frac{y-y'}{|\mathbf{r} - \mathbf{r}'|}\\ \frac{\partial}{\partial z}|\mathbf{r} - \mathbf{r}'| &= \frac{z-z'}{|\mathbf{r} - \mathbf{r}'|}\end{aligned}$$

We have

$$|\mathbf{r} - \mathbf{r}'|^2 = (x-x')^2 + (y-y')^2 + (z-z')^2,$$

so:

$$\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = \frac{-[(x-x')\,\hat{\mathbf{e}}_x + (y-y')\,\hat{\mathbf{e}}_y + (z-z')\,\hat{\mathbf{e}}_z]}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}$$

The denominators are clearly $|\mathbf{r} - \mathbf{r}'|^3$, thus

$$\begin{aligned}\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) &= -\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = -\frac{1}{|\mathbf{r} - \mathbf{r}'|^2}\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\\ &= -\frac{\hat{\mathbf{e}}_{r'r}}{|\mathbf{r} - \mathbf{r}'|^2}\end{aligned}$$

B

This follows from part A immediately by substitution. Remember ∇ is derivatives in terms of the *unprimed coordinates* $x,\ y$, and z; ∇ does *not* operate on $x',\ y'$, or z'.

C

$$\Phi(\mathbf{r}) = \int_{V'} \frac{\rho(\mathbf{r}')\,dV'}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|} = \int \frac{\lambda_0 a\,d\phi'}{4\pi\varepsilon_0(a^2+z^2)^{1/2}}$$

where we consider the infinitesimal charges $dq = (a\,d\phi)\lambda_0$ around the ring.

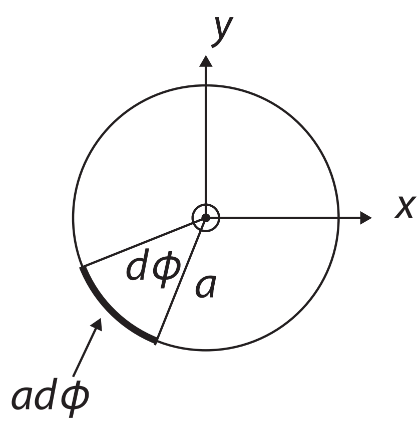

Figure 1: Diagram for Problem 3.1 Part C. Differential length $ad\phi$ in a circular hoop of line charge. (Image by MIT OpenCourseWare.)

We only care about the z-axis in the problem, so, by symmetry, there is no field in the x and y directions.

$$\Phi(\mathbf{r}) = \int_0^{2\pi} \frac{\lambda_0(a\,d\phi)}{4\pi\varepsilon_0(a^2+z^2)^{1/2}},$$

where $(a^2+z^2)^{1/2}$ is the distance from the charge $\lambda_0 a\,d\phi$ to the point z on the z-axis.

$$\boxed{\Phi(\mathbf{r}) = \frac{\lambda_0 a}{2\varepsilon_0(a^2+z^2)^{1/2}}} \text{ on the } z\text{-axis}$$

Check the limit as $z \to \infty$

$$\Phi(z\to\infty) = \frac{\lambda_0 a}{2\varepsilon_0|z|} = \frac{q_2}{4\pi\varepsilon_0|z|} \text{ (same form as point charge where } q_2 = \lambda_0 2\pi a) \checkmark$$

Now,

$$\mathbf{E} = -\nabla\Phi(\mathbf{r}) = -(\hat{\mathbf{e}}_x \cancelto{0}{\frac{\partial\Phi}{\partial x}} + \hat{\mathbf{e}}_y \cancelto{0}{\frac{\partial\Phi}{\partial y}} + \hat{\mathbf{e}}_z\frac{\partial\Phi}{\partial z}) = -\hat{\mathbf{e}}_z\frac{\partial}{\partial z}\left(\frac{\lambda_0 a}{2\varepsilon_0(a^2+z^2)^{1/2}}\right)$$

$$\boxed{\mathbf{E} = \hat{\mathbf{e}}_z\frac{a\lambda_0 z}{2\varepsilon_0(a^2+z^2)^{3/2}}}$$

Again, we check the limit as $z \to \infty$:

$$\mathbf{E}(z\to\infty) = \begin{cases} \hat{\mathbf{e}}_z\frac{\lambda_0 a}{2\varepsilon_0 z^2}; & z>0 \\ \hat{\mathbf{e}}_z\frac{-\lambda_0 a}{2\varepsilon_0 z^2}; & z<0 \end{cases} = \begin{cases} \hat{\mathbf{e}}_z\frac{q_2}{4\pi\varepsilon_0 z^2}; & z>0 \\ \hat{\mathbf{e}}_z\frac{-q_2}{4\pi\varepsilon_0 z^2}; & z<0 \end{cases} \quad \text{(same form as point charge)}$$

D

From part C

$$\Phi = \frac{\lambda_0 r}{2\varepsilon_0 (r^2 + z^2)^{1/2}}$$

for a ring of radius r. But now we have σ_0, not λ_0. How do we express λ_0 in terms of σ_0?

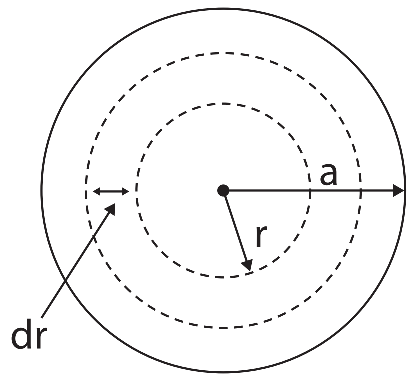

Figure 2: Diagram for Problem 3.1 Part D. Finding the scalar electric potential and electric field of a charged circular disk by adding up contributions from charged hoops of differential radial thickness. (Image by MIT OpenCourseWare.)

Take a ring of width dr in the disk (see figure). We have

$$\text{Total charge} = \underbrace{(r)(2\pi)}_{\text{circum.}}(dr)\sigma_0$$

$$\text{Line charge density} = \lambda_0 = \frac{\text{total charge}}{\text{length}} = \sigma_0 \, dr$$

So, $\lambda_0 = \sigma_0 \, dr$ and

$$d\Phi = \frac{\sigma_0 r \, dr}{2\varepsilon_0 (r^2 + z^2)^{1/2}}$$

Integrating gives

$$\Phi_{\text{total}} = \int_0^a \frac{\sigma_0 r \, dr}{2\varepsilon_0 (r^2 + z^2)^{1/2}} = \frac{\sigma_0}{2\varepsilon_0} \int_0^a \frac{r \, dr}{(r^2 + z^2)^{1/2}} = \frac{\sigma_0}{2\varepsilon_0} \left[\sqrt{r^2 + z^2}\right]_{r=0}^{r=a}$$

$$= \boxed{\frac{\sigma_0}{2\varepsilon_0} \left[\sqrt{a^2 + z^2} - |z|\right]}$$

$$\boxed{\mathbf{E} = -\nabla \Phi_{\text{total}} = \frac{\sigma_0 z}{2\varepsilon_0} \left[\frac{1}{|z|} - \frac{1}{\sqrt{a^2 + z^2}}\right] \hat{\mathbf{e}}_z}$$

As $a \to \infty$, z in $\sqrt{a^2 + z^2}$ can be neglected, so:

$$\left.\begin{array}{rl} \Phi_{\text{total}}(a \to \infty) & = -\frac{\sigma_0}{2\varepsilon_0}(z - a) \\ \mathbf{E}(a \to \infty) = -\nabla \Phi & = \hat{\mathbf{e}}_z \, \frac{\sigma_0}{2\varepsilon_0} \end{array}\right\} \; z > 0, \text{ just like sheet charge}$$

Problem 3.2

A

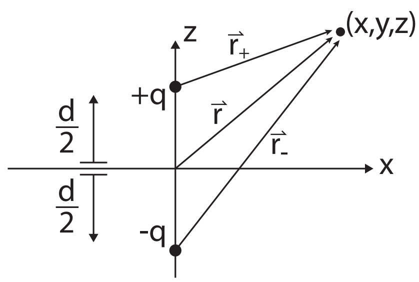

Figure 3: Diagram for Problem 3.2 Part A. (Image by MIT OpenCourseWare.)

We can simply add the potential contributions of each point charge:

$$\Phi = \frac{q}{4\pi\varepsilon_0 r_+} - \frac{q}{4\pi\varepsilon_0 r_-},$$

$$r_+ = \sqrt{x^2 + y^2 + \left(z - \frac{d}{2}\right)^2}$$

$$r_- = \sqrt{x^2 + y^2 + \left(z + \frac{d}{2}\right)^2}$$

$$\boxed{\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + \left(z - \frac{d}{2}\right)^2}} - \frac{1}{\sqrt{x^2 + y^2 + \left(z + \frac{d}{2}\right)^2}}\right]}$$

B

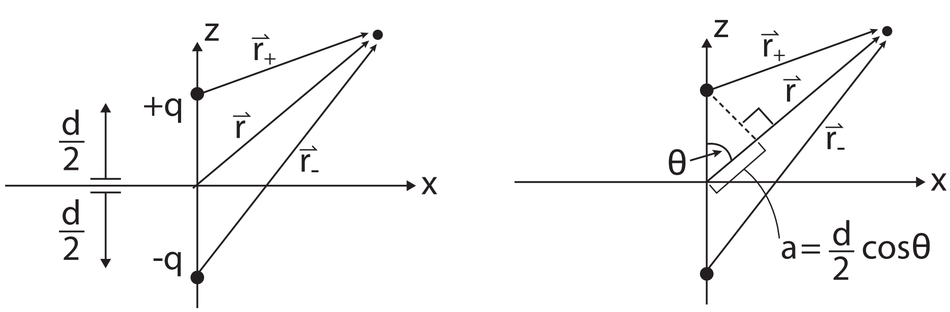

Figure 4: Diagrams for Problem 3.1 Part B. (Image by MIT OpenCourseWare.)

$p = qd$, where p is the dipole moment. We must make some approximations. As $r \to \infty$, $\mathbf{r}_+$, $\mathbf{r}_-$, and $\mathbf{r}$

become nearly parallel. Thus:

$$r_+ \approx r - a = r - \frac{d}{2}\cos\theta$$
$$r_+ \approx r\left(1 - \frac{d}{2r}\cos\theta\right).$$

Similarly,

$$r_- \approx r\left(1 + \frac{d}{2r}\cos\theta\right)$$

By part A,

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right].$$

If $|x| \ll 1$, then $1/(1+x) \approx 1 - x$. In addition,

$$\left|\frac{d}{2r}\cos\theta\right| \ll 1,$$

so

$$\frac{1}{r_+} \approx \frac{1}{r}\frac{1}{1 - \frac{d}{2r}\cos\theta} \approx \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right)$$
$$\frac{1}{r_-} \approx \frac{1}{r}\frac{1}{1 + \frac{d}{2r}\cos\theta} \approx \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)$$
$$\implies \frac{1}{r_+} - \frac{1}{r_-} \approx \frac{1}{r}\frac{d}{r}\cos\theta = \frac{d}{r^2}\cos\theta$$

$$\boxed{\Phi \approx \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}$$

C

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi = -\frac{\partial\Phi}{\partial r}\,\hat{\mathbf{e}}_r - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\,\hat{\mathbf{e}}_\theta - \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\,\hat{\mathbf{e}}_\phi$$
$$\frac{\partial\Phi}{\partial r} = -\frac{p\cos\theta}{2\pi\varepsilon_0 r^3}, \quad \frac{\partial\Phi}{\partial\theta} = -\frac{p\sin\theta}{4\pi\varepsilon_0 r^2}, \quad \frac{\partial\Phi}{\partial\phi} = 0$$
$$\mathbf{E} = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{p\sin\theta}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{e}}_\theta$$
$$\boxed{\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}[2\cos\theta\,\hat{\mathbf{e}}_r + \sin\theta\,\hat{\mathbf{e}}_\theta]}$$

D

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{E_r}{E_\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta$$
$$\frac{1}{r}dr = 2\cot\theta\,d\theta \implies \int\frac{1}{r}\,dr = \int 2\cot\theta\,d\theta$$
$$\ln r = 2\ln(\sin\theta) + k \implies r = r_0\sin^2\theta \text{ (when } \theta = \pi/2,\ r = r_0)$$
$$\boxed{\frac{r}{r_0} = \sin^2\theta}$$

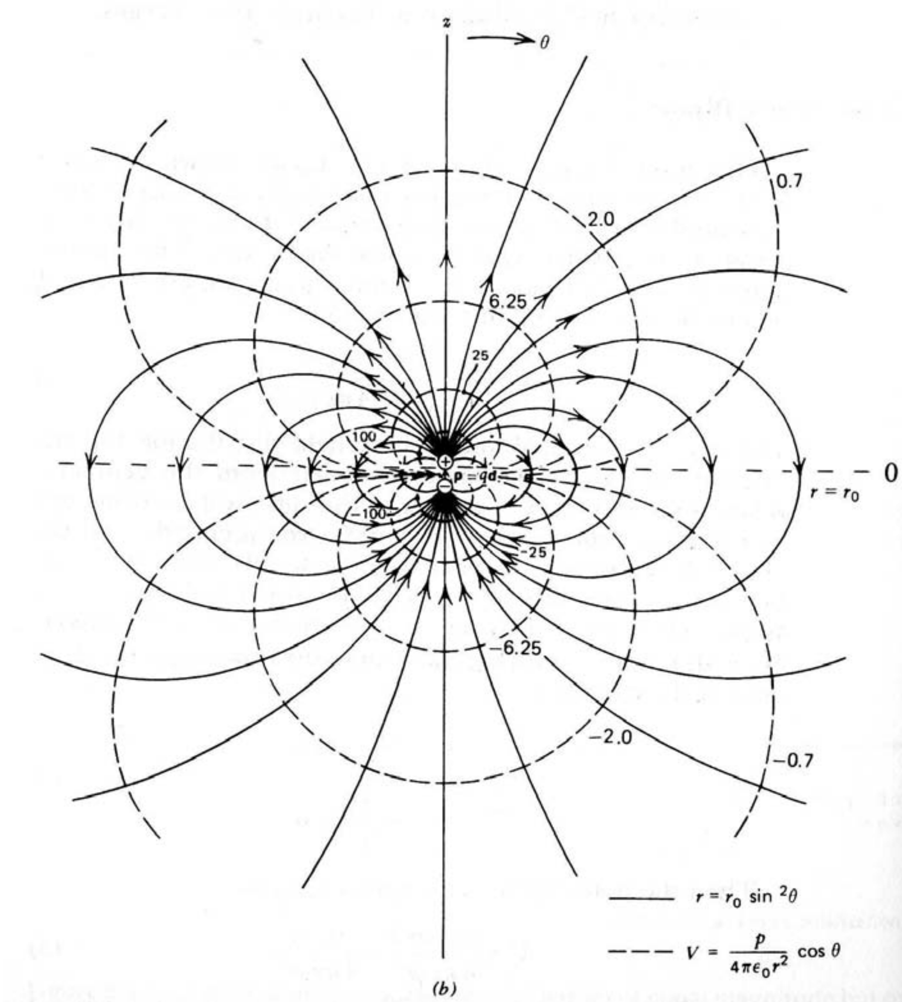

Figure 5: The potential at any point P due to the electric dipole is equal to the sum of potentials of each charge alone. The equi-potential (dashed) and field lines (solid) for a point electric dipole calibrated for $4\pi\varepsilon_0/p = 100$.

```
In[1]:= <<Graphics‘Graphics‘
In[2]:= r[ro_,theta_]:= ro*Sin[theta]^2
In[3]:= theta2 = Pi/2 - theta
In[4]:= eplot = PolarPlot[r[.25, theta2], r[.5, theta2], r[1, theta2], r[2, theta2]
          {theta, 0, 2*Pi}, PlotRange -> All]
```

Out[4]=

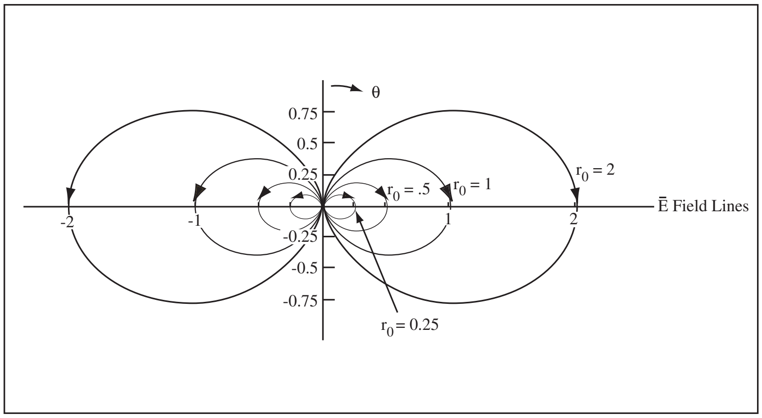

Figure 6: Mathematica Plot 1 – Electric field lines (Image by MIT OpenCourseWare.)

```
In[5]:= rp[phi_,theta_]:= Sqrt[Abs[Cos[theta]/(100*Phi)]]
In[6]:= pplot = PolarPlot[{rp[0.0025, theta2], rp[.01, theta2],
          rp[.04, theta2], rp[.16, theta2], rp[.64, theta2], rp[2.56, theta2],
          rp[10.24, theta2], rp[40.96, theta2]}, {theta, -Pi, Pi}, PlotRange -> All]
Out[6]=
```

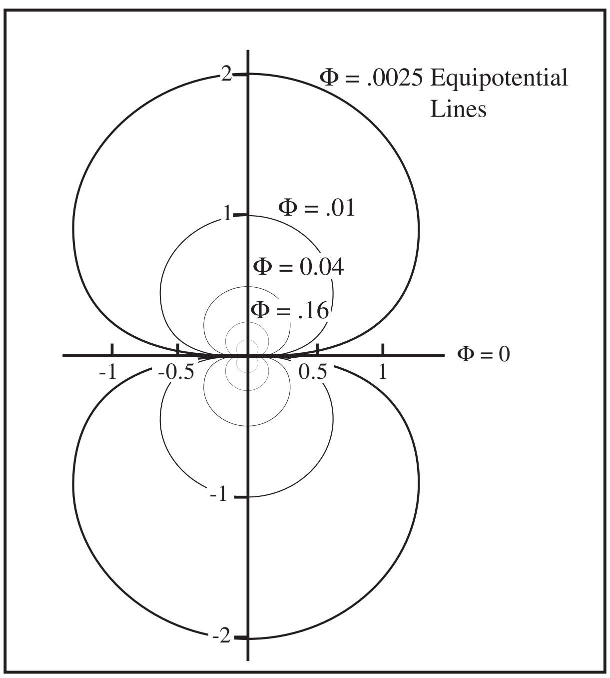

Figure 7: Mathematica Plot 2 – Equipotential lines (Image by MIT OpenCourseWare.)

```
In[7]:= tplot = Show[eplot, pplot]
```

Out[7]=

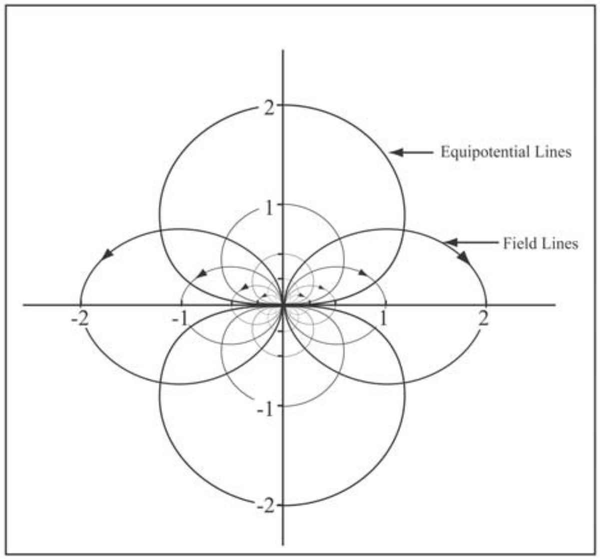

Figure 8: Mathematica Plot 3 – Electric field and equipotential Lines (Image by MIT OpenCourseWare.)

Problem 3.3

A

The bird acquires the same potential as the line, hence has charges induced on it and conserves charge when it flies away.

B

The fields are those of a charge Q at $y = h$, $x = Ut$ and an image at $y = -h$ and $x = Ut$.

C

The potential is the sum of that due to Q and its image $-Q$.

$$\Phi = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-Ut)^2+(y-h)^2+z^2}} - \frac{1}{\sqrt{(x-Ut)^2+(y+h)^2+z^2}}\right]$$

D

From this potential

$$E_y = -\frac{\partial\Phi}{\partial y} = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{y-h}{[(x-Ut)^2+(y-h)^2+z^2]^{3/2}} - \frac{y+h}{[(x-Ut)^2+(y-h)^2+z^2]^{3/2}}\right\}.$$

Thus, the surface charge density is

$$\sigma_0 = \varepsilon_0 E_y|_{y=0} = \frac{Q\varepsilon_0}{4\pi\varepsilon_0}\left[\frac{-h}{[(x-Ut)^2+h^2+z^2]^{3/2}} - \frac{h}{[(x-Ut)^2+h^2+z^2]^{3/2}}\right]$$
$$= \frac{-Qh}{2\pi[(x-Ut)^2+h^2+z^2]^{3/2}}$$

E

The net charge q on the electrode at any given instant is

$$q = \int_{z=0}^{w}\int_{x=0}^{l} \frac{-Qh\, dxdz}{2\pi[(x-Ut)^2+h^2+z^2]^{3/2}}.$$

If $w \ll h$,

$$q = \int_{x=0}^{l} \frac{-Qhw\, dx}{2\pi[(x-Ut)^2+h^2]^{3/2}}.$$

For the remaining integration, $x' = (x-Ut)$, $dx' = dx$, and

$$q = \int_{-Ut}^{l-Ut} \frac{-Qhw\, dx'}{2\pi[x'^2+h^2]^{3/2}}.$$

Thus,

$$\boxed{q = -\frac{Qw}{2\pi h}\left[\frac{l-Ut}{\sqrt{(l-Ut)^2+h^2}} + \frac{Ut}{\sqrt{(Ut)^2+h^2}}\right]}.$$

The dashed curves (1) and (2) in the figure 9(a) below are the first and second terms in the above equation. They sum to give (3).

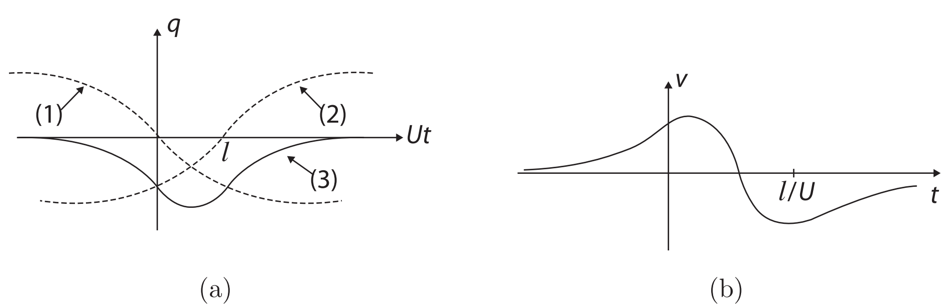

Figure 9: Curves for Problem 3.3 Part E. The net charge (a) and voltage (b) as a function of time on the electrode in the $y = 0$ plane. (Image by MIT OpenCourseWare.)

F

The current follows from the expression for q as

$$i = \frac{dq}{dt} = -\frac{Qw}{2\pi h}\left[\frac{-Uh^2}{[(l-Ut)^2+h^2]^{3/2}} + \frac{Uh^2}{[(Ut)^2+h^2]^{3/2}}\right]$$

and so the voltage is then $V = -iR = -R\, dq/dt$. A sketch is shown in figure 9(b) above.

Problem 3.4

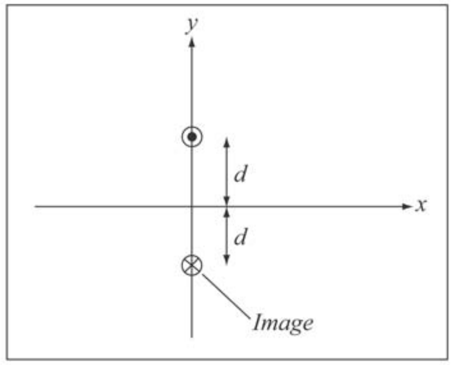

Figure 10: Diagram for Problem 3.4. The image current from a line current $I\hat{\mathbf{e}}_z$ a distance d above a perfect conductor. (Image by MIT OpenCourseWare.)

A

By the method of images, the image current is located at $(0, -d)$ with the current I in the opposite direction of the source current.
For a single line current I at the origin, the magnetic field is

$$\mathbf{H} = \frac{I}{2\pi r}\,\hat{\mathbf{e}}_\phi = \frac{I}{2\pi(x^2+y^2)}(-y\,\hat{\mathbf{e}}_x + x\,\hat{\mathbf{e}}_y).$$

Use the superposition for a current I in the $+z$ direction at $y = d$ so that y is replaced by $y - d$ and for the current $-I$ in the $-z$ direction at $y = -d$ so that y is replaced by $y + d$. Then

$$\mathbf{H}_{\text{total}} = \frac{I}{2\pi(x^2+(y-d)^2)}(-(y-d)\,\hat{\mathbf{e}}_x + x\,\hat{\mathbf{e}}_y) - \frac{I}{2\pi(x^2+(y+d)^2)}(-(y+d)\,\hat{\mathbf{e}}_x + x\,\hat{\mathbf{e}}_y)$$

B

The surface current at the $y = 0$ surface is

$$K_z = -H_x|_{y=0^+} \implies \mathbf{K} = \frac{-Id}{\pi(x^2+d^2)}\,\hat{\mathbf{e}}_z$$

C

The total current flowing on the $y = 0$ surface is

$$\mathbf{I}_{\text{total}} = \hat{\mathbf{e}}_z \int_{-\infty}^{+\infty} K_z\,dx = \frac{-Id\,\hat{\mathbf{e}}_z}{\pi}\int_{-\infty}^{+\infty}\frac{1}{(x^2+d^2)}\,dx = \frac{-Id\,\hat{\mathbf{e}}_z}{\pi}\frac{1}{d}\tan^{-1}\left(\frac{x}{d}\right)\Bigg|_{-\infty}^{+\infty} = -I\,\hat{\mathbf{e}}_z.$$

D

The force per unit length on the current I at $y = d$ comes from the image current at $y = -d$

$$\mathbf{F} = (I\,\hat{\mathbf{e}}_z) \times (\mu_0\mathbf{H}(x=0, y=d)) = \frac{\mu_0 I^2}{4\pi d}\,\hat{\mathbf{e}}_y.$$