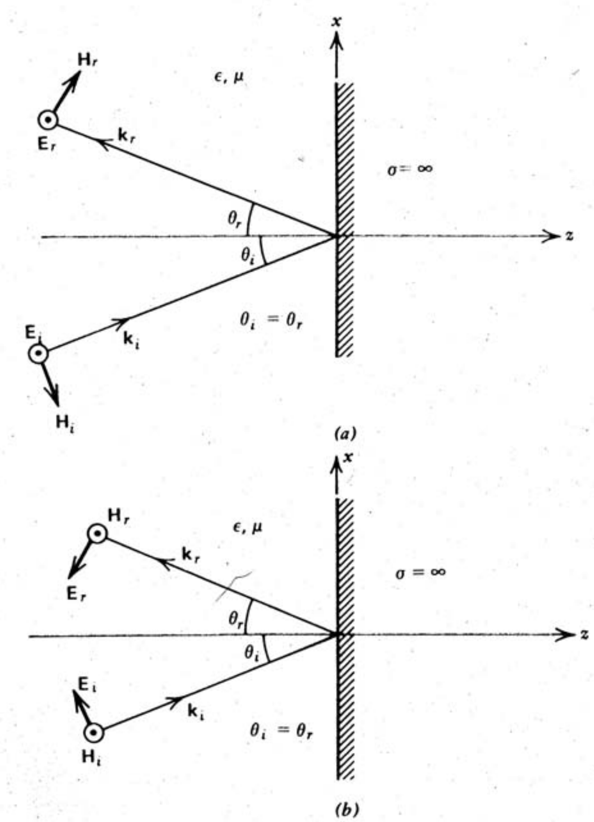
6.013 - Electromagnetics and Applications Fall 2005

Lecture 14 - Waveguides

Prof. Markus Zahn November 1, 2005

I. Parallel-Plate Waveguides

A. Waves with Oblique Incidence onto a Perfect Conductor



TE:

$$\bar{E} = 2E_i \sin(k_z z)\sin(\omega t - k_x x)\bar{i}_y$$

$$\bar{H} = \frac{2E_i}{\eta}\Big[-\cos(\theta_i)\cos(k_z z)\cos(\omega t - k_x x)\bar{i}_x + \sin(\theta_i)\sin(k_z z)\sin(\omega t - k_x x)\bar{i}_z\Big]$$

TM:

$$\bar{E} = 2E_i\Big[\cos(\theta_i)\sin(k_z z)\sin(\omega t - k_x x)\bar{i}_x - \sin(\theta_i)\cos(k_z z)\cos(\omega t - k_x x)\bar{i}_z\Big]$$

$$\bar{H} = \frac{2E_i}{\eta}\cos(k_z z)\cos(\omega t - k_x x)\bar{i}_y$$

$$k_x = k\sin(\theta_i), k_z = k\cos(\theta_i), k = \omega\sqrt{\epsilon\mu}, \eta = \sqrt{\frac{\mu}{\epsilon}}$$

Figure 7-17 A uniform plane wave obliquely incident upon a perfect conductor has its angle of incidence equal to the angle of reflection. (*a*) Electric field polarized parallel to the interface. (*b*) Magnetic field parallel to the interface.

B. Perfectly Conducting Plane Placed at $z = -d$

Boundary Conditions:

$$E_x(z=-d) = 0, E_y(z=-d) = 0, H_z(z=-d) = 0$$

$$\sin(k_z d) = 0 \Rightarrow k_z d = n\pi \Rightarrow k_z = \frac{n\pi}{d}$$

$$k_x^2 + k_z^2 = \omega^2\epsilon\mu \Rightarrow k_x = \sqrt{\omega^2\epsilon\mu - \left(\frac{n\pi}{d}\right)^2}$$

For wave propagation: k_x real $\Rightarrow \omega > \dfrac{n\pi c}{d}, c = \dfrac{1}{\sqrt{\epsilon\mu}}$

$$\text{Cutoff frequency } (n=1): \omega_{co} = \frac{\pi c}{d}$$

$$\text{Guide wavelength: } \lambda_x = \frac{2\pi}{k_x} = \frac{2\pi}{\sqrt{\omega^2 \epsilon\mu - \left(\frac{n\pi}{d}\right)^2}}$$

$$\text{Evanescent waves: } k_x^2 < 0 \Rightarrow \omega_n < \frac{n\pi c}{d}, c = \frac{1}{\sqrt{\epsilon\mu}}$$

$$k_x = j\alpha, \alpha = \sqrt{\left(\frac{n\pi}{d}\right)^2 - \omega^2\epsilon\mu} = \sqrt{\left(\frac{n\pi}{d}\right)^2 - \frac{\omega^2}{c^2}}$$

C. Time Average Power Flow

1. TE

$$\bar{E} = \text{Re}\left[\hat{\bar{E}}(z) e^{j(\omega t - k_x x)}\right]$$

$$\bar{H} = \text{Re}\left[\hat{\bar{H}}(z) e^{j(\omega t - k_x x)}\right]$$

$$\hat{\bar{E}}(z) = -2jE_i \sin(k_z z)\bar{i}_y$$

$$\hat{\bar{H}}(z) = \frac{2E_i}{\eta}\left[-\frac{k_z}{k}\cos(k_z z)\bar{i}_x + \frac{k_x}{k}(-j\sin(k_z z))\bar{i}_z\right]$$

$$\hat{\bar{S}} = \frac{1}{2}\hat{\bar{E}} \times \hat{\bar{H}}^* = \frac{-2j\cancel{(2)}}{\cancel{2}\eta}|E_i|^2 \sin(k_z z)\left[\frac{k_z}{k}\cos(k_z z)\bar{i}_z + \frac{jk_x^*}{k}\sin(k_z z)\bar{i}_x\right]$$

$$k = \omega\sqrt{\epsilon\mu} = \frac{\omega}{c}, k_x = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{n\pi}{d}\right)^2}, k_z = \frac{n\pi}{d}$$

$$\langle \bar{S} \rangle = \frac{1}{2}\text{Re}\left[\hat{\bar{E}} \times \hat{\bar{H}}^*\right] = \begin{cases} \frac{2|E_i|^2 k_x}{k\eta}\sin^2(k_z z)\bar{i}_x & \omega > \frac{n\pi c}{d} \\ 0 & \omega < \frac{n\pi c}{d} \end{cases}$$

$$\omega > \frac{n\pi c}{d} \Rightarrow k_x \text{ real} \qquad \text{(Propagating wave)}$$

$$\omega < \frac{n\pi c}{d} \Rightarrow k_x = j\alpha \quad (k_x \text{ imaginary: Evanescent wave})$$

2. TM

$$\hat{\bar{E}} = 2E_i\left[-\frac{jk_z}{k}\sin(k_z z)\bar{i}_x - \frac{k_x}{k}\cos(k_z z)\bar{i}_z\right]$$

$$\hat{\bar{H}} = \frac{2E_i}{\eta}\cos(k_z z)\bar{i}_y$$

$$\hat{\bar{S}} = \frac{1}{2}\left(\hat{\bar{E}} \times \hat{\bar{H}}^*\right) = \frac{2|E_i|^2\cos(k_z z)}{\eta k}\left[-jk_z\sin(k_z z)\bar{i}_z + k_x\cos(k_z z)\bar{i}_x\right]$$

3. Field-Line Plots, Surface Charge and Surface Current
   Surface Charge Distributions

   TE

$$\hat{\sigma}_s(z=0) = \hat{\sigma}_s(z=d) = 0$$

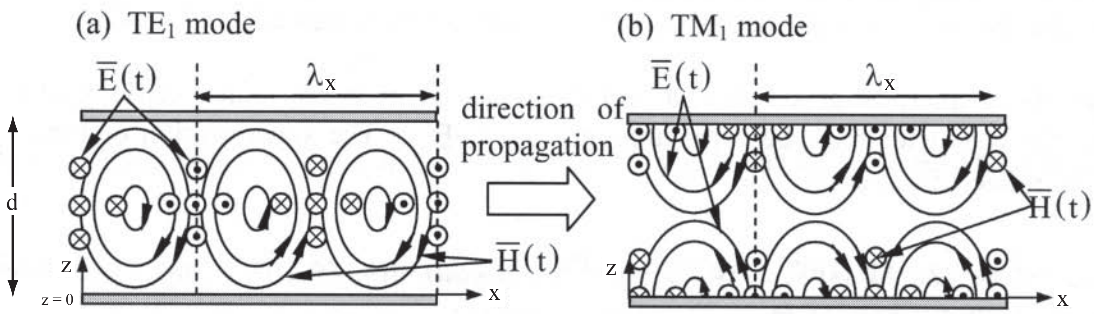


Figure 5.4.1 TE$_1$ and TM$_1$ modes of parallel-plate waveguides

TM

$$\hat{\sigma}_s(z=0) = \epsilon \hat{E}_z(z=0) = -2\epsilon E_i \frac{k_x}{k}$$
$$\hat{\sigma}_s(z=d) = -\epsilon \hat{E}_z(z=d) = +\frac{2\epsilon E_i k_x}{k}\cos(k_z d)$$
$$\cos(k_z d) = \cos(n\pi) = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

Surface Current Distributions

TE

$$\hat{K}_y(z=0) = \hat{H}_x(z=0) = -\frac{2E_i k_z}{\eta k}$$
$$\hat{K}_y(z=d) = -\hat{H}_x(z=d) = -\frac{2E_i k_z}{\eta k}\cos(k_z d)$$

TM

$$\hat{K}_x(z=0) = -\hat{H}_y(z=0) = -\frac{2E_i}{\eta}$$
$$\hat{K}_x(z=d) = \hat{H}_y(z=d) = \frac{2E_i}{\eta}\cos(k_z d)$$

II. Governing Equations

A. Maxwell's Equations in Linear Lossless Media with No Sources

$$\bar{J} = 0, \rho_f = 0, \bar{B} = \mu \bar{H}, \bar{D} = \epsilon \bar{E}$$

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t}$$
$$\nabla \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t}$$
$$\nabla \cdot \bar{E} = 0$$
$$\nabla \cdot \bar{H} = 0$$

B. Wave equations

$$\nabla \times (\nabla \times \bar{E}) = \nabla(\cancelto{0}{\nabla \cdot \bar{E}}) - \nabla^2 \bar{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \bar{H}) = -\epsilon\mu \frac{\partial^2 \bar{E}}{\partial t^2}$$
$$\nabla^2 \bar{E} = \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2}, c^2 = \frac{1}{\epsilon\mu}$$
$$\nabla \times (\nabla \times \bar{H}) = \nabla(\cancelto{0}{\nabla \cdot \bar{H}}) - \nabla^2 \bar{H} = \epsilon \frac{\partial}{\partial t}(\nabla \times \bar{E}) = -\epsilon\mu \frac{\partial^2 \bar{H}}{\partial t^2}$$
$$\nabla^2 \bar{H} = \frac{1}{c^2}\frac{\partial^2 \bar{H}}{\partial t^2}$$

III. Transverse Magnetic (TM) Modes ($H_z = 0$) [Rectangular Waveguide]

A. Solution for E_z

$$E_z = \mathrm{Re}\left[\hat{E}_z(x, y) e^{j(\omega t - k_z z)}\right]$$
$$\nabla^2 E_z = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} \Rightarrow \frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} - k_z^2 \hat{E}_z = -\frac{\omega^2}{c^2}\hat{E}_z$$
$$\frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k_z^2\right)\hat{E}_z = 0$$

Try product solution: $\hat{E}_z(x, y) = X(x)Y(y)$

$$Y(y)\frac{d^2 X(x)}{dx^2} + X(x)\frac{d^2 Y}{dy^2} = \left(k_z^2 - \frac{\omega^2}{c^2}\right) X(x)Y(y)$$
$$\underbrace{\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2}}_{-k_x^2} + \underbrace{\frac{1}{Y(y)}\frac{d^2 Y}{dy^2}}_{-k_y^2} = k_z^2 - \frac{\omega^2}{c^2}$$

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$
$$\frac{1}{X(x)}\frac{d^2 X}{dx^2} = -k_x^2 \Rightarrow \frac{d^2 X}{dx^2} + k_x^2 X(x) = 0$$
$$\frac{1}{Y(y)}\frac{d^2 Y}{dy^2} = -k_y^2 \Rightarrow \frac{d^2 Y}{dy^2} + k_y^2 Y(y) = 0$$

$$\begin{aligned} X(x) &= A_1\sin(k_x x) + A_2\cos(k_x x) \\ Y(y) &= B_1\sin(k_y y) + B_2\cos(k_y y) \end{aligned} \Rightarrow \begin{aligned} \hat{E}_z(x,y) &= X(x)Y(y) \\ &= (A_1\sin(k_x x) + A_2\cos(k_x x))\cdot \\ &\quad (B_1\sin(k_y y) + B_2\cos(k_y y)) \end{aligned}$$

B. Boundary Conditions

$$\hat{E}_z(x, y=0) = 0 \Rightarrow B_2 = 0$$

$$\Rightarrow \hat{E}_z(x,y) = E_0\sin(k_x x)\sin(k_y y)$$

$$\hat{E}_z(x=0, y) = 0 \Rightarrow A_2 = 0$$

$$\hat{E}_z(x, y=b) = 0 \Rightarrow k_y = \frac{n\pi}{b} \qquad n = 1, 2, 3, \ldots$$

$$\hat{E}_z(x=a, y) = 0 \Rightarrow k_x = \frac{m\pi}{a} \qquad m = 1, 2, 3, \ldots$$

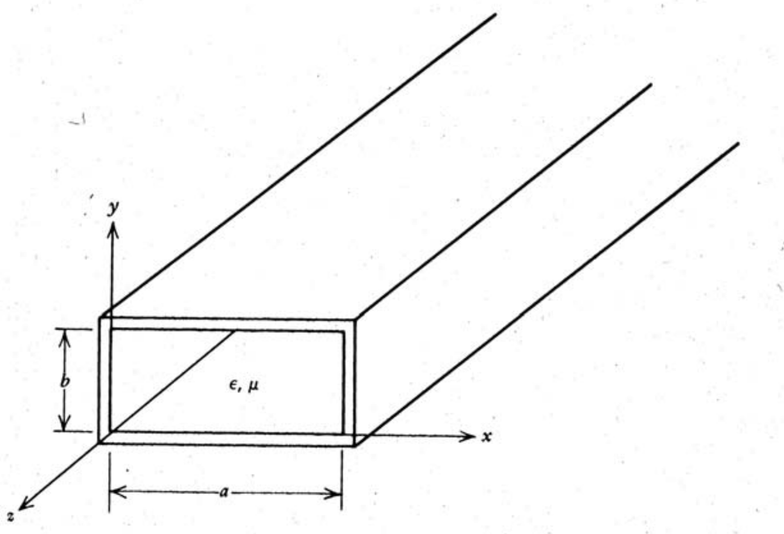


Figure 8-27 A lossless waveguide with rectangular cross section.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

C. Solution for E_x, E_y
TM_{mn} modes: $H_z = 0$

$$(\nabla \times \bar{E})_z = -\mu\frac{\partial H_z}{\partial t} = 0$$

$$\frac{\partial}{\partial x}\Bigg| \frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y}$$

$$\frac{\partial}{\partial y}\Bigg| \nabla \cdot \bar{E} = 0 = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{\partial^2 E_x}{\partial x \partial y}$$

$$\underbrace{\frac{\partial^2 E_x}{\partial x\partial y}}_{\frac{\partial^2 E_y}{\partial x^2}} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial y \partial z} = 0$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} = -\frac{\partial^2 E_z}{\partial y \partial z} \Rightarrow \frac{\partial^2 \hat{E}_y}{\partial x^2} + \frac{\partial^2 \hat{E}_y}{\partial y^2} = -\frac{\partial \hat{E}_z}{\partial y}(-jk_z)$$
$$= jk_z k_y E_0 \sin(k_x x)\cos(k_y y)$$

$$\hat{E}_y(x=0,y) = \hat{E}_y(x=a,y) = 0$$
$$\hat{E}_y = -\frac{jk_y k_z E_0}{k_x^2+k_y^2}\sin(k_x x)\cos(k_y y)$$
$$\frac{\partial \hat{E}_x}{\partial y} = \frac{\partial \hat{E}_y}{\partial x} = -\frac{jk_y k_z k_x}{k_x^2+k_y^2}E_0\cos(k_x x)\cos(k_y y)$$
$$\hat{E}_x = -\frac{jk_x k_z}{k_x^2+k_y^2}E_0\cos(k_x x)\sin(k_y y)$$

Check: $\hat{E}_x(x, y=0) = 0, \hat{E}_x(x, y=b) = 0$

D. Solution for $\bar{H}$ $\quad (\frac{\partial}{\partial z} \rightarrow -jk_z)$

$$\nabla \times \bar{E} = -\mu\frac{\partial \bar{H}}{\partial t} \Rightarrow \hat{H}_x = -\frac{1}{j\omega\mu}\left(\frac{\partial \hat{E}_z}{\partial y} + jk_z\hat{E}_y\right)$$
$$= \frac{j\omega\epsilon k_y}{k_x^2+k_y^2}E_0\sin(k_x x)\cos(k_y y)$$
$$\hat{H}_y = -\frac{1}{j\omega\mu}\left(-jk_z\hat{E}_x - \frac{\partial \hat{E}_z}{\partial x}\right)$$
$$= -\frac{j\omega\epsilon k_x}{k_x^2+k_y^2}E_0\cos(k_x x)\sin(k_y y)$$
$$\hat{H}_z = 0$$

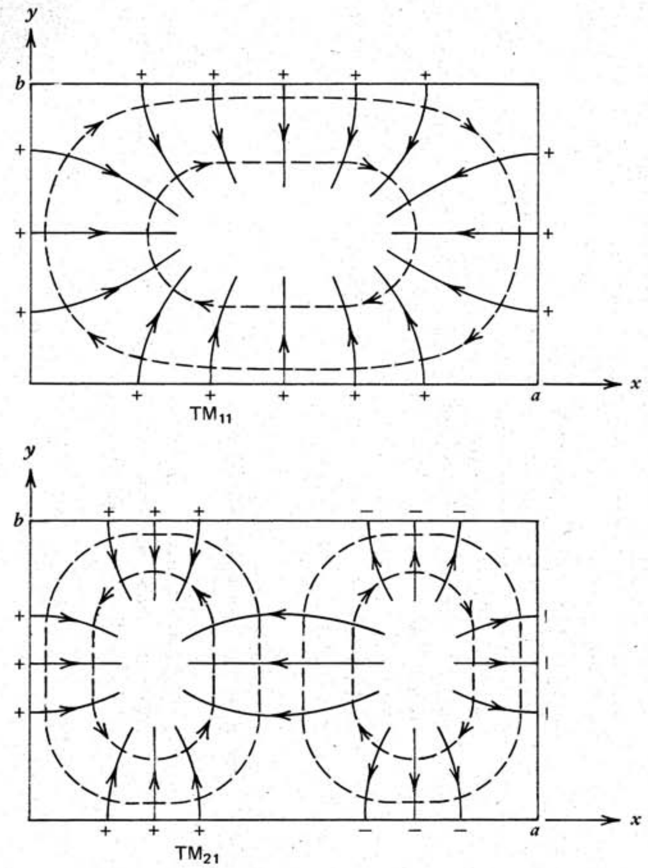
Check Boundary Conditions: $\hat{H}_y(x, y=0) = 0, \hat{H}_y(x=b, y) = 0$
$\hat{H}_x(x=0, y) = 0, \hat{H}_x(x=a, y) = 0$

E. Surface Charges and Currents

$$\hat{\sigma}_f(x=0,y) = \epsilon\hat{E}_x(x=0,y) = -\frac{jk_z k_x\epsilon}{k_x^2+k_y^2}E_0\sin(k_y y)$$
$$\hat{\sigma}_f(x=a,y) = -\epsilon\hat{E}_x(x=a,y) = \frac{jk_z k_x\epsilon}{k_x^2+k_y^2}E_0\cos(m\pi)\sin(k_y y)$$
$$\hat{\sigma}_f(x,y=0) = \epsilon\hat{E}_y(x,y=0) = -\frac{jk_z k_y\epsilon}{k_x^2+k_y^2}E_0\sin(k_x x)$$
$$\hat{\sigma}_f(x,y=b) = -\epsilon\hat{E}_y(x,y=b) = \frac{jk_z k_y\epsilon}{k_x^2+k_y^2}E_0\cos(n\pi)\sin(k_x x)$$
$$\hat{K}_z(x,y=0) = -\hat{H}_x(x,y=0) = -\frac{jk_y\omega\epsilon}{k_x^2+k_y^2}E_0\sin(k_x x)$$
$$\hat{K}_z(x,y=b) = \hat{H}_x(x,y=b) = \frac{jk_y\omega\epsilon}{k_x^2+k_y^2}E_0\cos(n\pi)\sin(k_x x)$$

$$\hat{K}_z(x=0,y) = \hat{H}_y(x=0,y) = -\frac{jk_x\omega\epsilon}{k_x^2+k_y^2}E_0\sin(k_y y)$$

$$\hat{K}_z(x=a,y) = -\hat{H}_y(x=a,y) = \frac{jk_x\omega\epsilon}{k_x^2+k_y^2}E_0\cos(m\pi)\sin(k_y y)$$



Electric field (———)

$$\hat{E}_x = \frac{-jk_xk_zE_0}{k_x^2+k_y^2}\cos k_x x \sin k_y y$$

$$\hat{E}_y = \frac{-jk_yk_zE_0}{k_x^2+k_y^2}\sin k_x x \cos k_y y$$

$$\hat{E}_z = E_0 \sin k_x x \sin k_y y$$

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{k_y}{k_x}\frac{\tan k_x x}{\tan k_y y}$$

$$\Rightarrow \frac{[\cos k_x x]^{(k_y/k_x)^2}}{\cos k_y y} = \text{const}$$

Magnetic field (– – – –)

$$\hat{H}_x = \frac{j\omega\varepsilon k_y}{k_x^2+k_y^2}E_0 \sin k_x x \sin k_y y$$

$$\hat{H}_y = \frac{-j\omega\varepsilon k_x}{k_x^2+k_y^2}E_0 \cos k_x x \sin k_y y$$

$$\frac{dy}{dx} = \frac{H_y}{H_x} = \frac{-k_x \cot k_x x}{k_y \cot k_y y}$$

$$\Rightarrow \sin k_x x \sin k_y y = \text{const}$$

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad k_z = \left[\frac{\omega^2}{c^2} - k_x^2 - k_y^2\right]^{1/2}$$

Figure 8-28 The transverse electric and magnetic field lines for the TM_{11} and TM_{21} modes. The electric field is purely z directed where the field lines converge.

IV. Transverse Electric (TE) Modes ($E_z = 0$) [Rectangular Waveguide]

A. Solution for H_z

$$H_z(x,y,z,t) = \text{Re}\left[\hat{H}_z(x,y)e^{j(\omega t - k_z z)}\right]$$

$$\nabla^2 H_z = \frac{1}{c^2}\frac{\partial^2 H_z}{\partial t^2} \Rightarrow \frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} - k_z^2\hat{H}_z = -\frac{\omega^2}{c^2}\hat{H}_z$$

$$\hat{H}_z(x,y) = (A_1\sin(k_x x) + A_2\cos(k_x x))(B_1\sin(k_y y) + B_2\cos(k_y y))$$

$$\text{with } k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} = \omega^2\epsilon\mu$$

Boundary Conditions:

$$\hat{H}_x(x=0,y) = 0, \hat{H}_x(x=a,y) = 0$$

$$\hat{H}_y(x,y=0) = 0, \hat{H}_y(x,y=b) = 0$$

B. Solutions for H_x, H_y

$$\hat{H}_x = \frac{jk_z k_x H_0}{k_x^2 + k_y^2}\sin(k_x x)\cos(k_y y) \qquad k_x = \frac{m\pi}{a}, k_y = \frac{n\pi}{b}$$

$$\hat{H}_y = \frac{jk_z k_y H_0}{k_x^2 + k_y^2}\cos(k_x x)\sin(k_y y) \qquad m = 0, 1, 2, 3, \ldots; n = 0, 1, 2, 3, \ldots$$

$$\hat{H}_z = H_0 \cos(k_x x)\cos(k_y y) \qquad \text{(but at least one of } m, n \text{ non-zero)}$$

C. Solutions for $\bar{E}$

$$\frac{\partial \bar{E}}{\partial t} = \frac{1}{\epsilon}\nabla \times \bar{H} \Rightarrow \frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\epsilon}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\hat{E}_x = \frac{1}{j\omega\epsilon}\left(\frac{\partial \hat{H}_z}{\partial y} + jk_z \hat{H}_y\right) = \frac{j\omega\mu k_y}{k_x^2 + k_y^2} H_0 \cos(k_x x)\sin(k_y y)$$

$$\hat{E}_y = \frac{1}{j\omega\epsilon}\left(-jk_z \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x}\right) = -\frac{j\omega\mu k_x}{k_x^2 + k_y^2} H_0 \sin(k_x x)\cos(k_y y)$$

$$\hat{E}_z = 0$$

D. Surface Charges and Currents

$$\hat{\sigma}_f(x = 0, y) = \epsilon \hat{E}_x(x = 0, y) = \frac{j\omega\epsilon\mu k_y}{k_x^2 + k_y^2} H_0 \sin(k_y y)$$

$$\hat{\sigma}_f(x = a, y) = -\epsilon \hat{E}_x(x = a, y) = -\frac{j\omega\epsilon\mu k_y}{k_x^2 + k_y^2} H_0 \sin(k_y y)\cos(m\pi)$$

$$\hat{\sigma}_f(x, y = 0) = \epsilon \hat{E}_y(x, y = 0) = -\frac{j\omega\epsilon\mu k_x}{k_x^2 + k_y^2} H_0 \sin(k_x x)$$

$$\hat{\sigma}_f(x, y = b) = -\epsilon \hat{E}_y(x, y = b) = \frac{j\omega\epsilon\mu k_x}{k_x^2 + k_y^2} H_0 \sin(k_x x)\cos(n\pi)$$

$$\hat{\bar{K}}(x = 0, y) = \bar{i}_x \times \left[\hat{\bar{H}}(x = 0, y)\right] = \bar{i}_z \hat{H}_y(x = 0, y) - \bar{i}_y \hat{H}_z(x = 0, y)$$

$$\hat{\bar{K}}(x = a, y) = -\bar{i}_x \times \left[\hat{\bar{H}}(x = a, y)\right] = -\bar{i}_z \hat{H}_y(x = a, y) + \bar{i}_y \hat{H}_z(x = a, y)$$

$$\hat{\bar{K}}(x, y = 0) = \bar{i}_y \times \left[\hat{\bar{H}}(x, y = 0)\right] = -\bar{i}_z \hat{H}_x(x, y = 0) + \bar{i}_x \hat{H}_z(x, y = 0)$$

$$\hat{\bar{K}}(x, y = b) = -\bar{i}_y \times \left[\hat{\bar{H}}(x, y = b)\right] = \bar{i}_z \hat{H}_x(x, y = b) - \bar{i}_x \hat{H}_z(x, y = b)$$

V. Cut-Off

$$k_x^2 + k_y^2 + k_z^2 = k_z^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \frac{\omega^2}{c^2}, c^2 = \frac{1}{\epsilon\mu}$$

$$k_z = \left[\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

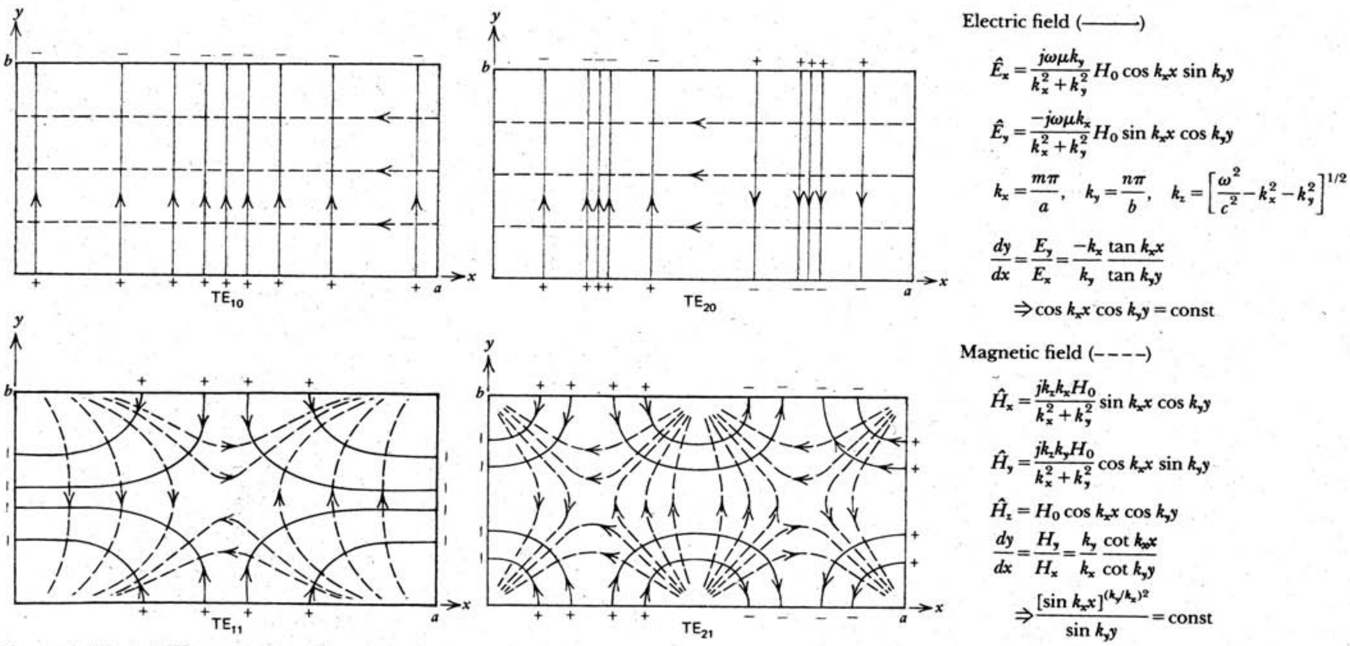
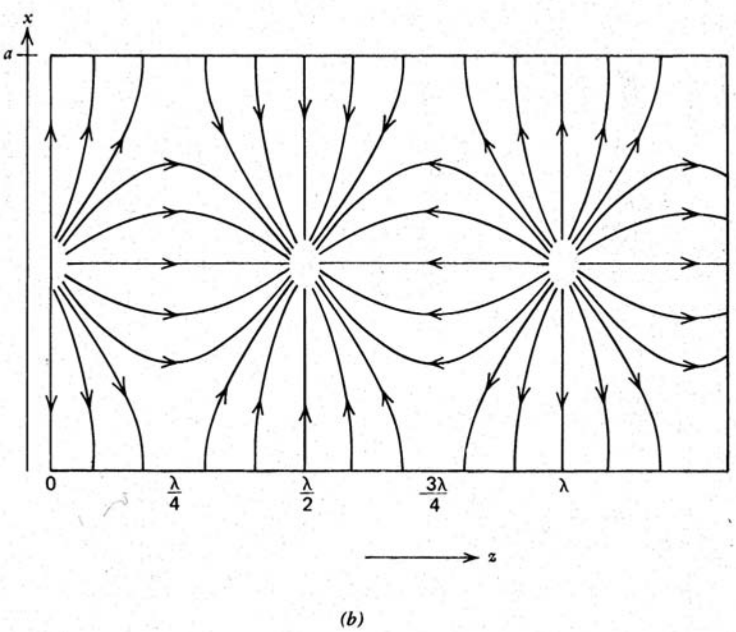


Figure 8-29 (*a*) The transverse electric and magnetic field lines for various TE modes. The magnetic field is purely *z* directed where the field lines converge. The TE$_{10}$ mode is called the dominant mode since it has the lowest cut-off frequency. (*b*) Surface current lines for the TE$_{10}$ mode.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.



(*b*)

Figure 8-29

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

Cut-off frequency: $k_z = 0 \Rightarrow \omega_{\text{co}} = c\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$

For $a > b$, the lowest cut-off frequency is for the TE$_{10}$ mode.

$$\omega_{\text{co}} = \frac{\pi c}{a} \Rightarrow f_{\text{co}} = \frac{\omega_{\text{co}}}{2\pi} = \frac{c}{2a}$$

For $a = 1$ cm, $c = 3 \times 10^8$ m/s $\Rightarrow f_{\text{co}} = \frac{3\times10^8}{2(.01)} = 1.5 \times 10^{10}$ Hz

For $a = 10$ m $\Rightarrow f_{\text{co}} = \frac{3\times10^8}{2(10)} = 15$ Mhz (Thus you cannot hear the radio in a tunnel.)

For $f < f_{co}, k_z$ is imaginary.

VI. Waveguide Power Flow

$$\langle \bar{S} \rangle = \frac{1}{2}\,\text{Re}\left[\hat{\bar{E}} \times \hat{\bar{H}}^*\right]$$

A. TM Modes

$$\begin{aligned}\langle \bar{S} \rangle &= \frac{1}{2}\,\text{Re}\left[e^{-jk_z z}\left(\hat{E}_x\bar{i}_x + \hat{E}_y\bar{i}_y + \hat{E}_z\bar{i}_z\right) \times \left(\hat{H}_x^*\bar{i}_x + \hat{H}_y^*\bar{i}_y\right)e^{+jk_z^* z}\right] \\ &= \frac{1}{2}\,\text{Re}\left[\left[\left(\hat{E}_x\hat{H}_y^* - \hat{E}_y\hat{H}_x^*\right)\bar{i}_z + \underbrace{\hat{E}_z\left(\hat{H}_x^*\bar{i}_y - \hat{H}_y^*\bar{i}_x\right)}_{\text{pure imaginary}}\right]e^{-j(k_z - k_z^*)z}\right]\end{aligned}$$

(k_z is imaginary below the cutoff)

$$\begin{aligned}e^{-j(k_z - k_z^*)z} &= \begin{cases} 1, f > f_{\text{co}} & k_z \text{ real} \\ e^{-2|k_z|z}, f < f_{\text{co}} & k_z \text{ imaginary}\end{cases} \\ \langle S_z \rangle &= \frac{\omega\epsilon|E_0|^2}{2(k_x^2 + k_y^2)}\,\text{Re}\left[k_z e^{-j(k_z - k_z^*)z}\left(k_x^2\cos^2(k_x x)\sin^2(k_y y) + k_y^2\sin^2(k_x x)\cos^2(k_y y)\right)\right] \\ &= \begin{cases} 0 & k_z \text{ imaginary } (f < f_{\text{co}}) \\ \frac{\omega\epsilon|E_0|^2 k_z}{2(k_x^2 + k_y^2)}\left(k_x^2\cos^2(k_x x)\sin^2(k_y y) + k_y^2\sin^2(k_x x)\cos^2(k_y y)\right) & k_z \text{ real } (f > f_{\text{co}})\end{cases} \\ \langle P \rangle &= \int_{x=0}^{a}\int_{y=0}^{b} \langle S_z \rangle \, dx\, dy \\ &= \frac{\omega\epsilon k_z ab E_0^2}{8(k_x^2 + k_y^2)} \qquad k_z \text{ real } (f > f_{co})\end{aligned}$$

For TM Modes, $m, n = 1, 2, 3, \ldots$ ($m = 0$ or $n = 0$ not allowed)

B. TE Modes

$$\begin{aligned}\langle \bar{S} \rangle &= \frac{1}{2}\,\text{Re}\left[\left(\hat{E}_x\bar{i}_x + \hat{E}_y\bar{i}_y\right)e^{-jk_z z} \times \left(\hat{H}_x^*\bar{i}_x + \hat{H}_y^*\bar{i}_y + \hat{H}_z^*\bar{i}_z\right)e^{jk_z^* z}\right] \\ &= \frac{1}{2}\,\text{Re}\left[\left[\left(\hat{E}_x\hat{H}_y^* - \hat{E}_y\hat{H}_x^*\right)\bar{i}_z - \underbrace{\hat{H}_z^*\left(\hat{E}_x\bar{i}_y - \hat{E}_y\bar{i}_x\right)}_{\text{pure imaginary}}\right]e^{-j(k_z - k_z^*)z}\right]\end{aligned}$$

$$\begin{aligned}\langle S_z \rangle &= \frac{1}{2}\frac{\omega\mu|H_0|^2}{k_x^2 + k_y^2}\left(k_y^2\cos^2(k_x x)\sin^2(k_y y) + k_x^2\sin^2(k_x x)\cos^2(k_y y)\right) \times \text{Re}\left[k_z e^{-j(k_z - k_z^*)z}\right] \\ &= \begin{cases} 0 & k_z \text{ imaginary } (f < f_{\text{co}}) \\ \frac{1}{2}\frac{\omega\mu|H_0|^2}{k_x^2 + k_y^2}k_z\left(k_y^2\cos^2(k_x x)\sin^2(k_y y) + k_x^2\sin^2(k_x x)\cos^2(k_y y)\right) & k_z \text{ real } (f > f_{\text{co}})\end{cases} \\ \langle P \rangle &= \int_{x=0}^{a}\int_{y=0}^{b} \langle S_z \rangle \, dx\, dy = \begin{cases} \frac{\omega\mu k_z ab|H_0|^2}{8(k_x^2 + k_y^2)} & m, n \neq 0 \\ \frac{\omega\mu k_z ab|H_0|^2}{4(k_x^2 + k_y^2)} & m \text{ or } n = 0\end{cases}\end{aligned}$$