REVIEW OF UPW BASICS

Example: $\hat{x}$-polarized UPW traveling in $+\hat{z}$ direction

$$\overline{E} = \hat{x}E_o \cos(\omega t - kz)$$

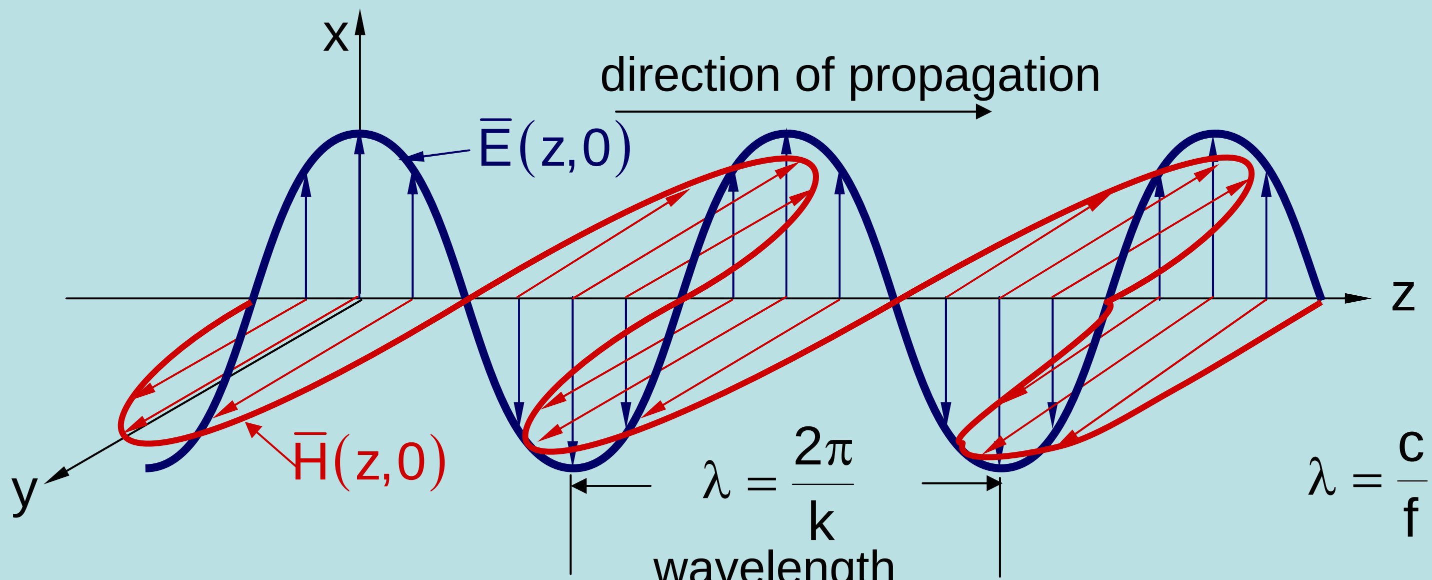
$$\underline{\overline{E}}(z) = \hat{x}\underline{E}_o e^{-jkz}$$

$$\overline{H} = \hat{y}\frac{E_o}{\eta_o}\cos(\omega t - kz)$$

$$\underline{\overline{H}}(z) = \hat{y}\frac{\underline{E}_o}{\eta_o}e^{-jkz}$$

$\overline{E} \times \overline{H}$: $\hat{z}$ direction of propagation

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = 3x10^8 \, m/s \quad \omega(rads/s) = 2\pi f \quad k(rads/m) = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad \eta_o = \sqrt{\frac{\mu_o}{\varepsilon_0}}$$



HOW DO WAVES CONVEY POWER, ENERGY?

Recall: $\overline{E}$ [V/m] • $\overline{J}$ [A/m^2] $= P_d$ [W/m^3]   But $\overline{E} \perp \overline{H}$

Manipulate Ampere's law to get $\overline{E} \bullet \overline{J}$

$\overline{E} \cdot (\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t})$ For symmetry, compute $\overline{H} \cdot (\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t})$

$$\underbrace{\overline{H} \cdot \left(\nabla \times \overline{E}\right) - \overline{E} \cdot \left(\nabla \times \overline{H}\right)}_{\text{Vector Identity}} = -\overline{H} \cdot \left(\frac{\partial \overline{B}}{\partial t}\right) - \overline{E} \cdot \left(\overline{J} + \cdot \frac{\partial \overline{D}}{\partial t}\right)$$

$$\nabla \cdot \left(\overline{E} \times \overline{H}\right) = -\overline{H} \cdot \frac{\partial \overline{B}}{\partial t} - \overline{E} \cdot \overline{J} - \overline{E} \cdot \frac{\partial \overline{D}}{\partial t} \quad (\text{W/m}^3)$$

This is **Poynting's Theorem**   What does it mean?

POYNTING THEOREM

Poynting's Theorem: $\nabla \cdot (\bar{E} \times \bar{H}) = -\bar{H} \cdot \frac{\partial \bar{B}}{\partial t} - \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} - \bar{E} \cdot \bar{J}$

$\bar{B} = \mu \bar{H} \qquad \bar{D} = \varepsilon \bar{E}$

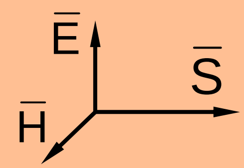
$$\underbrace{\nabla \cdot (\bar{E} \times \bar{H})} = -\frac{d}{dt}\underbrace{\left(\frac{1}{2}\mu |\bar{H}|^2\right)} - \frac{d}{dt}\underbrace{\left(\frac{1}{2}\varepsilon |\bar{E}|^2\right)} - \underbrace{(\bar{E} \cdot \bar{J})} \quad (W/m^3)$$

- Poynting vector, $\bar{S}$ [W/m²]
- Stored magnetic energy density, W_m
- Stored electric energy density, W_e
- Power density dissipated/m³, W_d

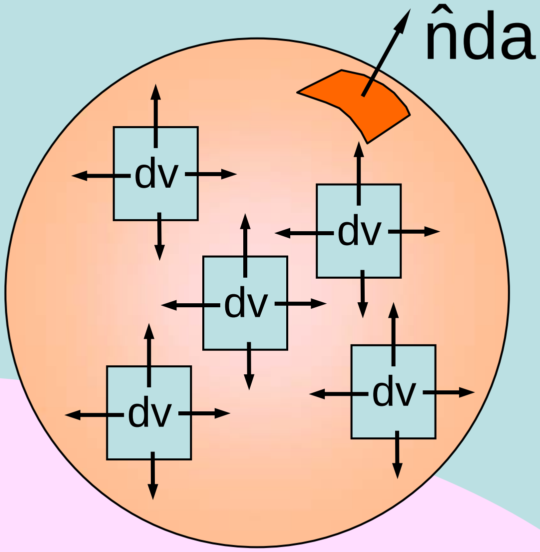
Energy is conserved!

Note: $\frac{d}{dt}\left(\frac{1}{2}\mu |\bar{H}|^2\right) = \mu |\bar{H}| \frac{d|\bar{H}|}{dt} = \bar{H} \bullet \frac{d\bar{B}}{dt}$

Poynting Vector $\bar{S} = \bar{E} \times \bar{H} \ (W/m^2)$ $\left(\frac{\text{volts}}{\text{m}} \cdot \frac{\text{amps}}{\text{m}} = \frac{\text{watts}}{\text{m}^2}\right)$



INTEGRAL POYNTING THEOREM



Use: $\oint_S \bar{A}\cdot\hat{n}\, da = \int_V \nabla\cdot\bar{A}\, dv$

Gauss's Theorem (not Gauss's Law)

Therefore:

$$\oint_S (\bar{E}\times\bar{H})\cdot\hat{n}\, da = \int_V \nabla\cdot(\bar{E}\times\bar{H})\, dv$$

$$= \int_V \left[-\frac{d}{dt}\left(\frac{1}{2}\mu|\bar{H}|^2\right) - \frac{d}{dt}\left(\frac{1}{2}\varepsilon|\bar{E}|^2\right) - (\bar{E}\cdot\bar{J}) \right] dv$$

$$\oint_S (\bar{E}\times\bar{H})\cdot\hat{n}\, da = -\int_V \frac{d}{dt}\left(\frac{1}{2}\varepsilon|\bar{E}|^2 + \frac{1}{2}\mu|\bar{H}|^2\right) dv - \int_V \bar{E}\cdot\bar{J}\, dv$$

Power emerging = released stored energy - dissipation [W]

The Poynting vector $\triangleq\ \bar{S} = \bar{E}\times\bar{H}$ gives both the magnitude of the power density (intensity) and the direction of its flow.

UNIFORM PLANE WAVE EXAMPLE

$$\bar{E} = \hat{x} E_o \cos(\omega t - kz)$$

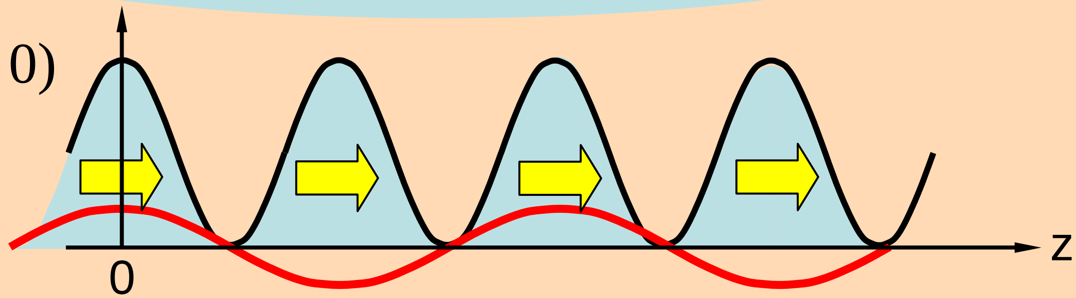
$$\bar{H} = \hat{y}\left(\frac{E_o}{\eta_o}\right)\cos(\omega t - kz)$$

$$W_e = \frac{1}{2}\varepsilon_o E_o^2 \cos^2(\omega t - kz)$$

$$W_m = \frac{1}{2}\frac{\mu_o}{\eta_o^2} E_o^2 \cos^2(\omega t - kz)$$

$$\bar{S}(t) = \bar{E} \times \bar{H} = \hat{z}\left(\frac{E_o^2}{\eta_o}\right)\cos^2(\omega t - kz) \quad (\mathrm{W/m^2})$$

$\bar{S}(z \text{ at } t = 0)$



$$\bar{S} = \hat{z}\frac{E_o^2}{\eta_0}\cos^2(\omega t - kz) \Rightarrow \boxed{\langle \bar{S} \rangle = \hat{z}\frac{1}{2}\frac{E_o^2}{\eta_o} = I(\theta, \phi, r) \quad [\mathrm{w/m^2}]}$$

The time average $\langle \bar{S}(r, \theta, \phi) \rangle$ is "intensity" [W/m²]

COMPLEX NOTATION – POYNTING VECTOR

Defining a meaningful $\underline{\bar{S}}$ and relating it to $\bar{S}$ is not obvious.

Let's work backwards to find the time average $\langle \bar{S} \rangle$ and then $\underline{\bar{S}}$

$$\bar{S}(t) = \bar{E} \times \bar{H} = \mathrm{Re}\left[\underline{\bar{E}} \cdot e^{j\omega t}\right] \times \mathrm{Re}\left[\underline{\bar{H}} \cdot e^{j\omega t}\right]$$

$$= \underbrace{[\bar{E}_r \cos(\omega t) - \bar{E}_i \sin(\omega t)]}_{} \times \underbrace{[\bar{H}_r \cos(\omega t) - \bar{H}_i \sin(\omega t)]}_{}$$

$$\Rightarrow \langle \bar{S}(t) \rangle = \frac{1}{2}\left[\left(\bar{E}_r \times \bar{H}_r\right) + \left(\bar{E}_i \times \bar{H}_i\right)\right]$$

$$= \frac{1}{2} R_e \underbrace{\left(\underline{\bar{E}} \times \underline{\bar{H}}^*\right)}_{\underline{\bar{S}} \text{ (by definition}} \quad [= \frac{1}{2} R_e\{(E_r + jE_i) \times (H_r - jH_i)\}]$$

Thus, we can define $\langle \bar{S} \rangle = \frac{1}{2} \mathrm{Re}\left(\underline{\bar{E}} \times \underline{\bar{H}}^*\right)$ and $\underline{\bar{S}} = \underline{\bar{E}} \times \underline{\bar{H}}^*$

Recall: $\underline{\bar{E}} = \bar{E}_r + j\bar{E}_i \qquad \underline{\bar{H}} = \bar{H}_r + j\bar{H}_i \qquad e^{j\omega t} = \cos \omega t + j \sin \omega t$

UPW REFLECTED BY PERFECT CONDUCTOR

$\bar{E} = \hat{x}E_+ \cos(\omega t - kz) + \hat{x}E_- \cos(\omega t + kz)$
$= 0$ at $z = 0$ (perfect conductor)

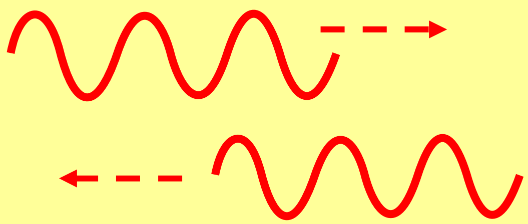
} Forward plus reflected wave

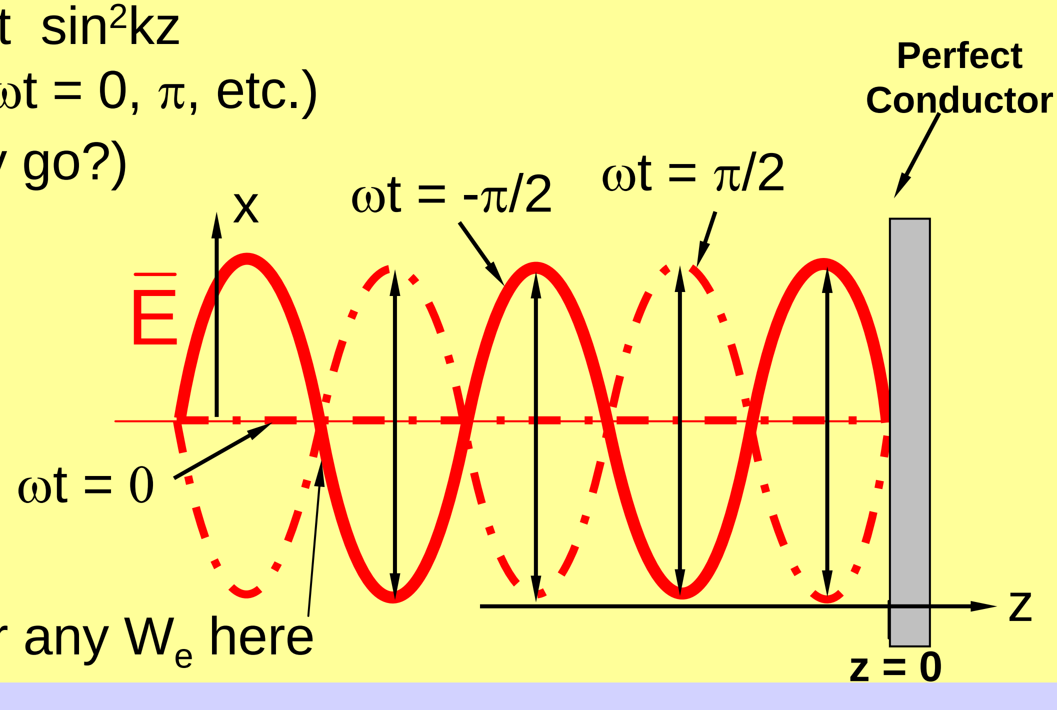
$\Rightarrow E_- = -E_+$ (Solving for unknown reflection)

$\Rightarrow \bar{E} = \hat{x}2E_+ \sin\omega t \cdot \sin kz$ Standing waves, oscillate without moving

(recall: $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2} \bullet \sin\frac{\alpha-\beta}{2}$)

W_e [J/m³] $= 2\,\varepsilon E^2 \sin^2\omega t \sin^2 kz$
E = 0 every half cycle ($\omega t = 0, \pi$, etc.)
(Where does the energy go?)





STANDING WAVE EXAMPLE - CONTINUED

(It's in the H field!)

$$\bar{E} = \hat{x}\,[E_+ \cos(\omega t - kz) + E_- \cos(\omega t + kz)]$$

$$\bar{H} = \hat{y}\,[\frac{E_+}{\eta_o}\cos(\omega t - kz) - \frac{E_-}{\eta_o} y \cos(\omega t + kz)]$$

Note

$$E_- = -E_+ \Rightarrow \quad \bar{H} = \hat{y}\frac{2E_+}{\eta_o}\cos\omega t \cdot \cos kz$$

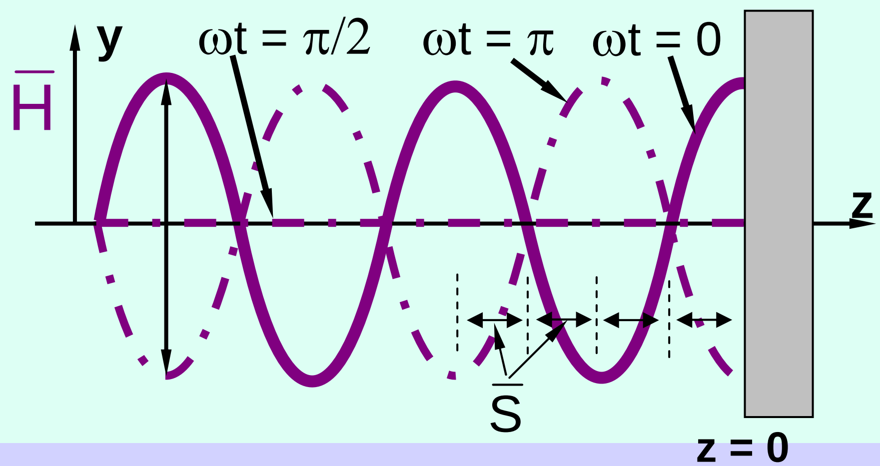
$$W_m[J/m^3] = \frac{1}{2}\mu_o\left(\frac{2E_+}{\eta_o}\right)^2 \cos^2\omega t \cos^2 kz = 2\varepsilon E_+^2 \cos^2\omega t \cos^2 kz$$

$= 0$ when $\omega t = \pi/2, 3\pi/2$, etc.)

$$\bar{S} = \bar{E} \times \bar{H} = \hat{z}\; 4\frac{E_+^2}{\eta_o}\cos\omega t \sin\omega t \cdot \cos kz \sin kz$$

$$= \hat{z}\frac{E_+^2}{\eta_o}\cdot \sin 2kz \cdot \sin 2\omega t$$

$$\Rightarrow \langle \bar{S} \rangle = 0 = \frac{1}{2}\text{Re}\left(\underline{\bar{E}} \times \underline{\bar{H}}^*\right)$$



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