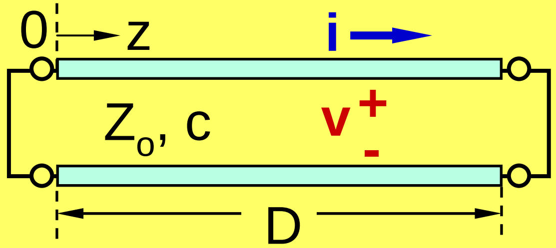
TEM RESONATORS

Voltages and Currents:



$$v(t,z) = V_+ \cos(\omega t - kz) + V_- \cos(\omega t + kz)$$

Boundary conditions: $v(t,z) = 0$ at $z = 0 \quad \Rightarrow \quad V_- = -V_+$

$$v(t,z) = V_+ [\cos(\omega t - kz) - \cos(\omega t + kz)] = 2V_+ \sin \omega t \sin kz \; *$$

$$i(t,z) = \left(\frac{V_+}{Z_o}\right)[\cos(\omega t - kz) + \cos(\omega t + kz)] = 2\left(\frac{V_+}{Z_o}\right)\cos \omega t \cos kz \; *$$

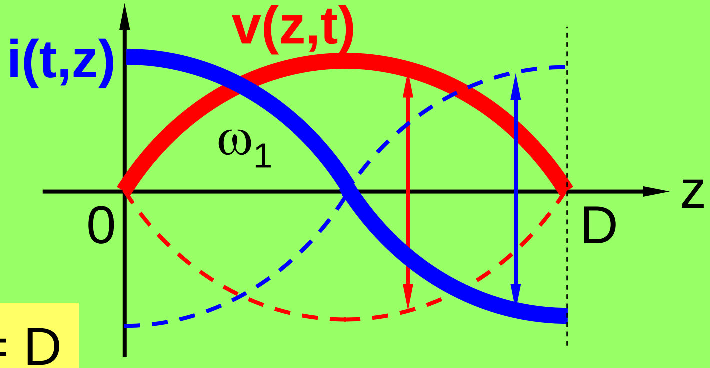
Resonant Frequencies ω_n:

$2V_+ \sin(\omega t) \sin(kD) = 0 \quad \Rightarrow$

$k_n D = n\pi, \quad n = 0, 1, 2, \ldots$

$k_n D = (\omega_n / c) D = n\pi \quad \Rightarrow$

$\omega_n = n\pi c/D = 2\pi c/\lambda_n \quad \Rightarrow \quad n\lambda_n/2 = D$



$$* \; \cos\alpha - \cos\beta = -2 \sin[(\alpha + \beta)/2] \sin[(\alpha - \beta)/2]$$

$$\cos\alpha + \cos\beta = 2 \cos[(\alpha + \beta)/2] \cos[(\alpha - \beta)/2]$$

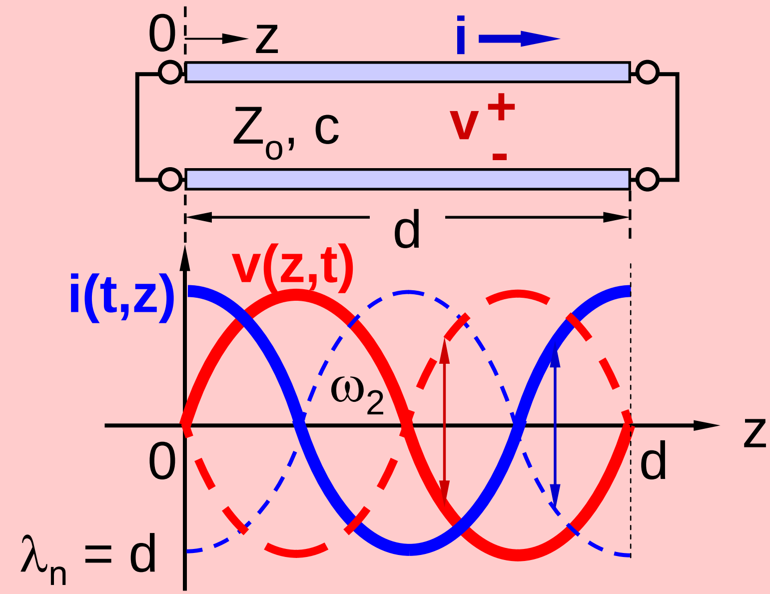
MORE TEM RESONANCES

Resonant Frequencies ω_n:

$$\omega_n = \frac{n\pi c}{d} \qquad \frac{n\lambda_n}{2} = d$$

DC Resonance ($\omega = 0$):

Current flows in loop,
Voltages are zero
$w_e = 0$, $w_m \neq 0$ [J]



Open-Circuit Resonator:

$n\lambda_n/2 = d$, $n = 0,1,2,\ldots \Rightarrow$

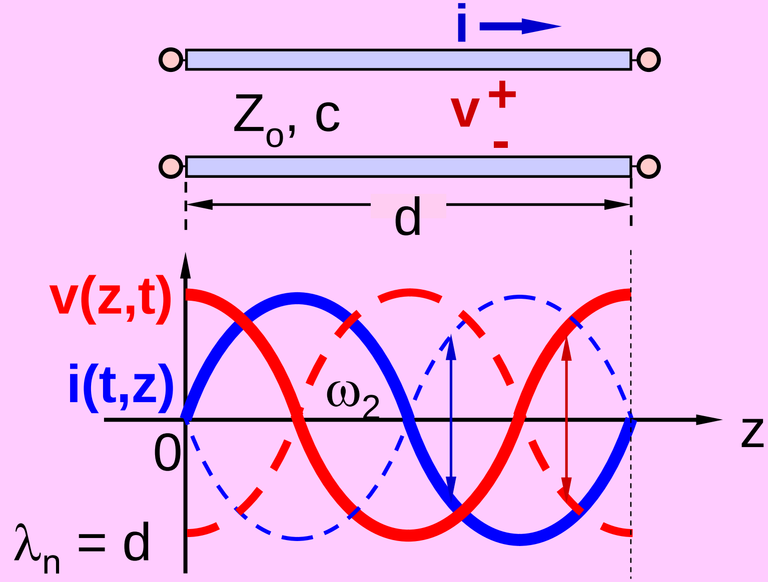
$\lambda_n = 2d/n$ $\qquad$ $\omega_n = n\pi c/d = 2\pi c/\lambda_n$

$\lambda_0 = \infty \rightarrow \omega_o = 0$

$\lambda_1 = 2d \rightarrow \omega_1 = \pi c/d$

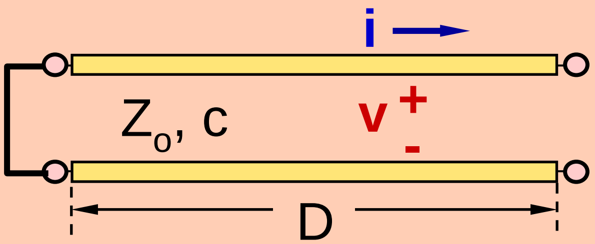
$\lambda_2 = d \rightarrow \omega_2 = 2\pi c/d$

$\lambda_3 = 2d/3 \rightarrow \omega_3 = 3\pi c/d$



ANOTHER TEM RESONANCE

Quarter-Wave Resonator:



$D = \lambda/4,\ 3\lambda/4,\ 5\lambda/4, \ldots \Rightarrow$

$D = (\lambda_n/4)(2n - 1)\quad [n = 1,2,3,\ldots]$

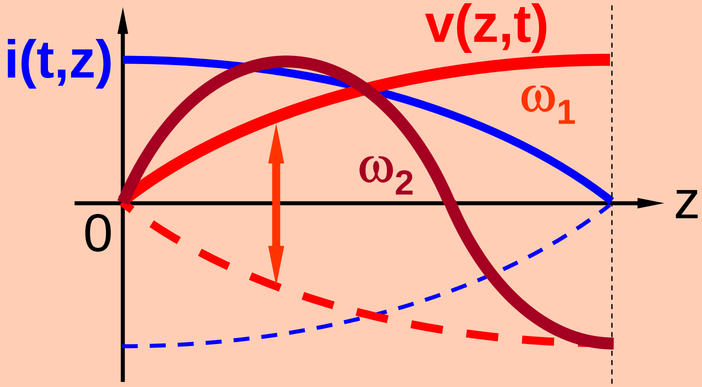
$\lambda_n = 4D/(2n - 1)$

$\lambda_1 = 4D$

$\lambda_2 = 4D/3$

$\lambda_3 = 4D/5$

$\lambda_4 = 4D/7$



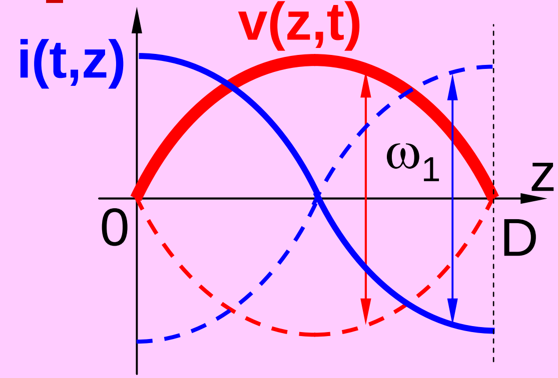
ENERGY IN TEM RESONATORS

Electric Energy Density $W_e = Cv^2/2$ [J/m]:

$$v(t,z) = V_o \sin\omega_n t \sin k_n z$$

$$W_e(t,z) = \frac{1}{2}CV_0^2(\sin^2\omega_n t)(\sin^2 k_n z) \quad [\text{J/m}]$$

$$= \frac{1}{2}CV_o^2(1-\cos 2\omega_n t)(1-\cos 2k_n z) \geq 0$$

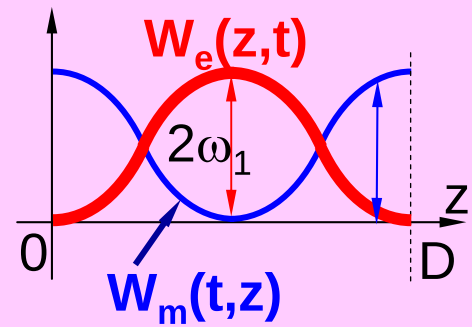


Magnetic Energy Density $W_m = Li^2/2$:

$$i(t,z) = \frac{V_o}{Z_o}\cos\omega_n t \cos k_n z$$

$$W_m(t,z) = \frac{1}{2}L\left(\frac{V_o}{Z_o}\right)^2\cos^2\omega_n t \cos^2 k_n z \quad [\text{J/m}]$$

$$= \frac{1}{2}L\left(\frac{V_o}{Z_o}\right)^2(1+\cos 2\omega_n t)(1+\cos 2k_n z)$$



Total Electric and Magnetic Energies w_e and w_m:

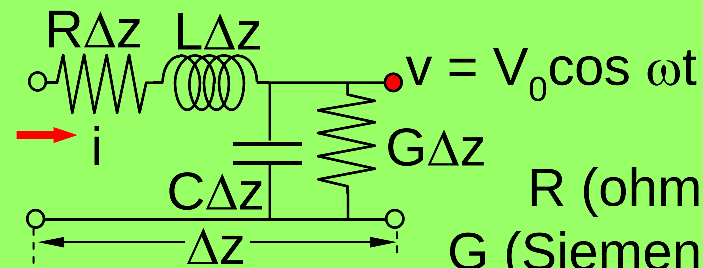
$$\langle W_e\rangle = \frac{1}{4}CV_o^2(1-\cos 2k_n z) \qquad \langle W_m\rangle = \frac{1}{4}L\left(\frac{V_o}{Z_o}\right)^2(1+\cos 2k_n z)$$

Over $n\frac{\lambda}{4}$ we have $\langle w_e\rangle = \langle w_m\rangle$ because $CV_o^2 = L\left(\frac{V_o}{Z_o}\right)^2$ (recall $Z_o^2 = \frac{L}{C}$)

* < • > signifies time average of •.

TEM RESONATOR LOSS AND Q

Distributed Losses:



R (ohms m^{-1})

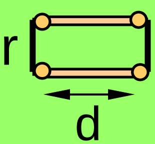
G (Siemens m^{-1})

$$P_{diss}(t,z) = i^2R + v^2G \quad [\text{watts}/\text{m}]$$

$$P_d\,[W] = \int_0^D \langle P_{diss}\rangle dz$$

$$= \int_0^D (V_o/Z_o)^2 \langle \cos^2\omega_n t\rangle (\cos^2 k_n z) R dz$$

$$+ \int_0^D V_o^{\,2} \langle \sin^2\omega_n t\rangle (\sin^2 k_n z) G dz$$

(for )

$$P_d = \frac{RD}{4}(V_o/Z_o)^2 + \frac{GDV_o^{\,2}}{4} \quad [\text{watts}]$$

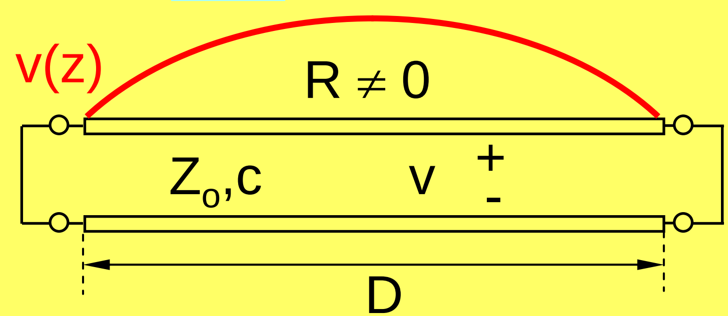
Example of Q Calculation (n =1):

$$Q_I = Q_L = \frac{\omega_o w_T}{P_d} = \frac{\pi c C Z_o^{\,2}}{RD} = \pi \frac{1}{\sqrt{LC}} \frac{C(L/C)}{RD} = \frac{\pi Z_o}{RD}$$

$$\omega_o(n=1) = \frac{\pi c}{D}$$

$$w_T = 2\langle w_e\rangle = \frac{1}{4} C V_o^2 D$$

$$P_d = \frac{RD}{4}\left(\frac{V_o}{Z_o}\right)^2$$



LUMPED LOSSES IN TEM RESONATORS

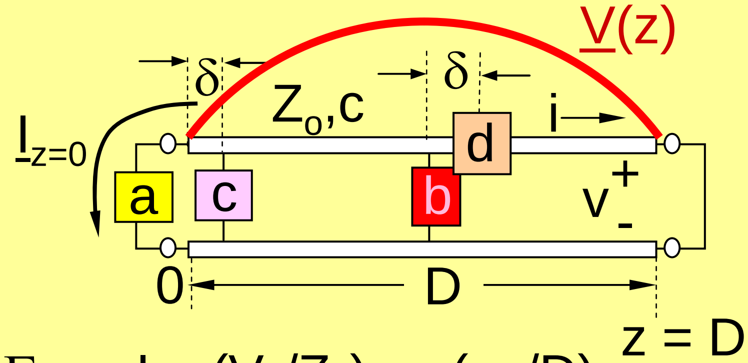
Calculation of Q:

$$Q = \frac{\omega_o w_T}{P_d}$$

Perturbation Technique:

Find P_d using unperturbed v or i

Need $(\Delta v/v_o) << 1$, $(\Delta i/i_o) << 1$, $\Gamma \cong \Gamma_o$



$\underline{I} = (\underline{V}_o/Z_o)\cos(\pi z/D)$

Loss Computation Examples:

Use unperturbed $\underline{V}$, $\underline{I}$

a $P_d \cong \frac{1}{2} |\underline{I}_{z=0}|^2 R_a$

c $P_d \cong \frac{1}{2R_c} |\underline{V}_o \sin k\delta|^2$

b $P_d \cong \frac{1}{2R_b} |\underline{V}_{z=d/2}|^2$

d $P_d \cong \frac{1}{2} |(\underline{V}_o/Z_o)\sin k\delta|^2 R_d$

When is a perturbation too large?

a Want $\Gamma_{z=0} \cong -1 \cong \frac{Z_n - 1}{Z_n + 1}$ $\Rightarrow$ $R_a << Z_o$ $(Z_{an} << 1)$

b Want $\Gamma_{(z=D/2)} \cong 1$ $\Rightarrow$ $R_b >> Z_o$ $(Z_{bn} >> 1)$

c Want $R_c >> |-jZ_o \sin k\delta|$ $\Rightarrow$ $R_c > \sim Z_o$ (if $2\pi\delta/\lambda << 1$)

COUPLED TEM RESONATORS

Example: computation of Q:

$$Q_{int,ext,or\ loaded} = \frac{\omega_o w_T}{P_{d:int,ext,or\ loaded}}$$

$$\omega_o = \frac{\pi c}{D} \qquad w_T = 2<w_e> = 2D\frac{C|V_o|^2}{8}$$

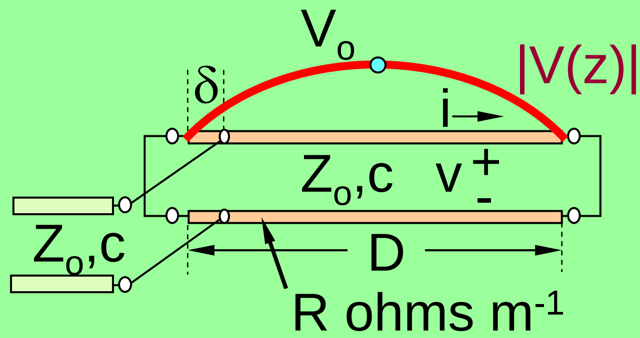
$$P_{d;ext} = \frac{\left|V_o \sin(\frac{\pi\delta}{D})\right|^2}{2Z_o} \Rightarrow Q_{ext} \cong \frac{(\frac{D}{\delta})^2}{2\pi}$$

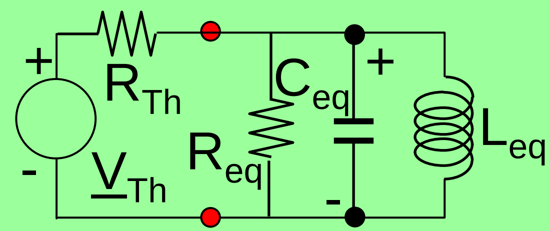
$$P_{d;int} = \frac{RD}{4}\left|\frac{V_o}{Z_o}\right|^2 \Rightarrow Q_{int} = \frac{\pi Z_o}{RD}$$

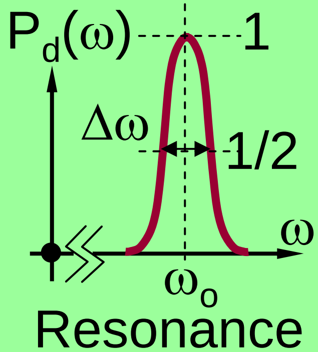
Example, perfect coupling:

$$Q_{int} = Q_{ext} \Rightarrow \delta = \sqrt{\frac{RD^3}{2\pi^2 Z_o}}$$

$$\Delta\omega = \frac{\omega_o}{Q_L} = \frac{2\omega_o}{Q_I} = 2\frac{R}{L} = \frac{R_{eq}}{L_{eq}} [r/s]$$







COUPLED TEM RESONATORS

Resonator equivalent circuit

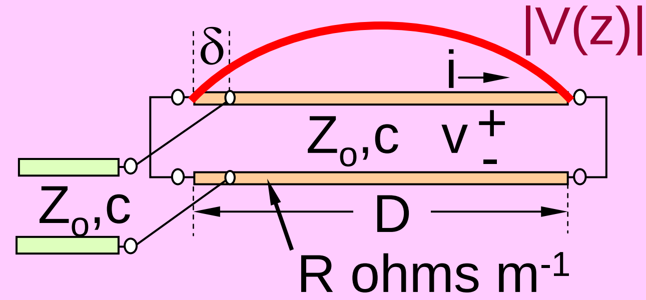
$$\Delta\omega = \frac{\omega_o}{Q_L} = \frac{2\omega_o}{Q_I} = \frac{R_{eq}}{L_{eq}} \quad [r/s]$$

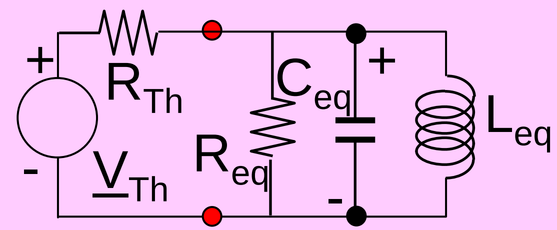
$$\omega_o = \frac{\pi c}{D} = \frac{1}{\sqrt{L_{eq}C_{eq}}}$$

$$Z_n(\omega = \omega_o) = \frac{R_{eq}}{R_{Th}} = \frac{P_{d;ext}}{P_{d;int}} = \frac{Q_I}{Q_E}$$

3 Eqn
3 Unk

$$\Rightarrow \Gamma \text{ at } \omega_o \quad \left(\Gamma = \frac{R_{eq}-R_{Th}}{R_{eq}+R_{Th}}\right)$$



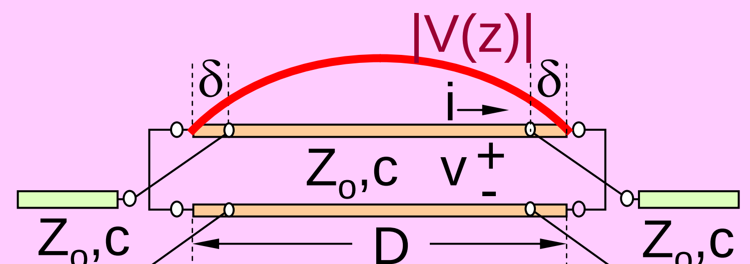


Imperfect coupling if $Q_E \neq Q_I$

Lossless bandpass filter

If both lines are Z_o and $\delta_1 = \delta_2$,

Then $Q_I = Q_E \Rightarrow$ perfect coupling and perfect match at ω_o



Lossless bandpass filter