Final Solutions:
December 15, 2009

Problem 2. (20 points)

(a) (5 points)
We're given that the joint PDF is constant in the shaded region, and since the PDF must integrate to 1, we know that the constant must equal 1 over the area of the region. Thus,

$$c = \frac{1}{1/2} = 2.$$

(b) (5 points)
The marginal PDFs of X and Y are found by integrating the joint PDF over all possible y's and x's, respectively. To find the marginal PDF of X, we take a particular value x and integrate over all possible y values in that vertical "slice" at $X = x$. Since the joint PDF is constant, this integral simplifies to just multiplying the joint PDF by the width of the "slice". Because the width of the slice is always $1/2$ for any $x \in [0, 1]$, we have that the marginal PDF of X is uniform over that interval:

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since the joint PDF is symmetric, the marginal PDF of Y is also uniform:

$$f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) (5 points)
To find the conditional expectation and variance, first we need to determine what the conditional distribution is given $Y = 1/4$. At $Y = 1/4$, we take a horizontal slice of a uniform joint PDF, which gives us a uniform distribution over the interval $x \in [1/4, 3/4]$. Thus, we have

$$\mathbf{E}[X \mid Y = 1/4] = \frac{1}{2},$$

$$\text{var}(X \mid Y = 1/4) = \frac{(1/2)^2}{12} = \frac{1}{48}.$$

(d) (5 points)
At $Y = 3/4$, we have a horizontal slice of the joint PDF, which is nonzero when $x \in [0, 1/4] \cup [3/4, 1]$. Since the joint PDF is uniform, the slice will also be uniform, but only in the range of x where the joint PDF is nonzero (i.e. where (x, y) lies in the shaded region). Thus, the conditional PDF of X is

$$f_{X|Y}(x \mid 3/4) = \begin{cases} 2, & x \in [0, 1/4] \cup [3/4, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Problem 3. (25 points)

(a) (5 points)
The recurrent states are {3,4}.

(b) (5 points)
The 2-step transition probability from State 2 to State 4 can be found by enumerating all the possible sequences. They are $\{2 \to 1 \to 4\}$ and $\{2 \to 4 \to 4\}$. Thus,

$$\mathbf{P}(X_2 = 4 \mid X_0 = 2) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot 1 = \frac{7}{18}.$$

(c) (5 points)
Generally,

$$r_{11}(n+1) = \sum_{j=1}^{4} p_{1j} r_{j1}(n).$$

Since states 3 and 4 are absorbing states, this expression simplifies to

$$r_{11}(n+1) = \frac{1}{4} r_{11}(n) + \frac{1}{4} r_{21}(n).$$

Alternatively,

$$\begin{aligned} r_{11}(n+1) &= \sum_{k=1}^{4} r_{1k}(n) p_{k1} \\ &= r_{11}(n) \cdot \frac{1}{4} + r_{12}(n) \cdot \frac{1}{3}. \end{aligned}$$

(d) (5 points)
The steady-state probabilities do not exist since there is more than one recurrent class. The long-term state probabilities would depend on the initial state.

(e) (5 points)
To find the probability of being absorbed by state 4, we set up the absorption probabilities. Note that $a_4 = 1$ and $a_3 = 0$.

$$\begin{aligned} a_1 &= \frac{1}{4} a_1 + \frac{1}{4} a_2 + \frac{1}{3} a_3 + \frac{1}{6} a_4 \\ &= \frac{1}{4} a_1 + \frac{1}{4} a_2 + \frac{1}{6} \end{aligned}$$

$$\begin{aligned} a_2 &= \frac{1}{3} a_1 + \frac{1}{3} a_3 + \frac{1}{3} a_4 \\ &= \frac{1}{3} a_1 + \frac{1}{3} \end{aligned}$$

Solving these equations yields $a_1 = \frac{3}{8}$.

Problem 4. (30 points)

(a) (5 points)
Given the problem statement, we can treat Al, Bonnie, and Clyde's running as 3 independent Poisson processes, where the arrivals correspond to lap completions and the arrival rates indicate the number of laps completed per hour. Since the three processes are independent, we can merge them to create a new process that captures the lap completions of all three runners. This merged process will have arrival rate $\lambda_M = \lambda_A + \lambda_B + \lambda_C = 68$. The total number of completed laps, L, over the first hour is then described by a Poisson PMF with $\lambda_M = 68$ and $\tau = 1$:

$$p_L(\ell) = \begin{cases} \frac{68^\ell e^{-68}}{\ell!}, & \ell = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(b) (5 points)
Let L be the total number of completed laps over the first hour, and let C_i be the number of cups of water consumed at the end of the ith lap. Then, the total number of cups of water consumed is

$$C = \sum_{i=1}^{L} C_i,$$

which is a sum of a random number of i.i.d. random variables. Thus, we can use the law of iterated expectations to find

$$\mathbf{E}[C] = \mathbf{E}[\mathbf{E}[C \mid L]] = \mathbf{E}[LC_i] = \mathbf{E}[L]\mathbf{E}[C_i] = (\lambda_M \tau) \cdot \left(1 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3}\right) = 68 \cdot \frac{5}{3} = \frac{340}{3}.$$

(c) (5 points)
Let X be the number of laps (out of 72) after which Al drank 2 cups of water. Then, in order for him to drink at least 130 cups, we must have

$$1 \cdot (72 - X) + 2 \cdot X \geq 130,$$

which implies that we need

$$X \geq 58.$$

Now, let X_i be i.i.d. Bernoulli random variables that equal 1 if Al drank 2 cups of water following his ith lap and 0 if he drank 1 cup. Then

$$X = X_1 + X_2 + \cdots + X_{72}.$$

X is evidently a binomial random variable with $n = 72$ and $p = 2/3$, and the probability we are looking for is

$$\mathbf{P}(X \geq 58) = \sum_{k=58}^{72} \binom{72}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{72-k}.$$

This expression is difficult to calculate, but since we're dealing with the sum of a relatively large number of i.i.d. random variables, we can invoke the Central Limit Theorem to approximate this probability using a normal distribution. In particular, we can approximate X as

a normal random variable with mean $np = 72 \cdot 2/3 = 48$ and variance $np(1-p) = 16$ and approximate the desired probability as

$$\mathbf{P}(X \geq 58) = 1 - \mathbf{P}(X < 58) \approx 1 - \Phi\left(\frac{58-48}{\sqrt{16}}\right) = 1 - \Phi(2.5) \approx 0.0062.$$

(d) (5 points)
The event that Al is the first to finish a lap is the same as the event that the first arrival in the merged process came from Al's process. This probability is

$$\frac{\lambda_A}{\lambda_A + \lambda_B + \lambda_C} = \frac{21}{68}.$$

(e) (5 points)
This is an instance of the random incidence paradox, so the duration of Al's current lap consists of the sum of the duration from the time of your arrival until Al's next lap completion and the duration from the time of your arrival back to the time of Al's previous lap completion. This is the sum of 2 independent exponential random variables with parameter $\lambda_A = 21$ (i.e. a second- order Erlang random variable):

$$f_T(t) = \begin{cases} 21^2 t e^{-21t}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(f) (5 points)
As in the previous part, the duration of Al's second lap consists of the time remaining from $t = 1/4$ until he completes his second lap and the time elapsed since he began his second lap until $t = 1/4$. Let X be the time elapsed and Y be the time remaining. We can still model the time remaining Y as an exponential random variable. However, we can no longer do the same for the time elapsed X because we know X can be no larger than $1/4$, whereas the exponential random variable can be arbitrarily large.

To find the PDF of X, let's first consider its CDF.

$$\begin{aligned}
\mathbf{P}(X \leq x) &= \mathbf{P}(\text{The 1 arrival occurred less than } x \text{ hours ago from time } 1/4) \\
&= \frac{\mathbf{P}(\text{1 arrival in the interval } [1/4 - x, 1/4] \text{ and no arrivals in the interval } [0, 1/4 - x])}{\mathbf{P}(\text{1 arrival in the interval } [0, 1/4])} \\
&= \frac{\mathbf{P}(\text{1 arrival in the interval } [1/4 - x, 1/4])\mathbf{P}(\text{no arrivals in the interval } [0, 1/4 - x])}{\mathbf{P}(\text{1 arrival in the interval } [0, 1/4])} \\
&= \frac{\mathbf{P}(1, x)\mathbf{P}(0, 1/4 - x)}{\mathbf{P}(1, 1/4)} \\
&= \frac{e^{-21x}(21x)e^{-21(1/4-x)}}{e^{-21/4}(21/4)} \\
&= \begin{cases} 0, & x < 0, \\ 4x, & x \in [0, 1/4], \\ 1, & x > 1/4. \end{cases}
\end{aligned}$$

Thus, we find that the X is uniform over the interval $[0, 1/4]$, with PDF

$$f_X(x) = \begin{cases} 4, & x \in [0, 1/4], \\ 0, & \text{otherwise.} \end{cases}$$

The total time that Al spends on his second lap is $T = X + Y$. Since X and Y correspond to disjoint time intervals in the Poisson process, they are independent, and therefore we can use convolution to find the PDF of T:

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\, dx \\ &= \int_0^{\min(1/4,t)} 4 \cdot 21 e^{-21(t-x)}\, dx \\ &= \begin{cases} 4e^{-21t}\left(e^{21\min(1/4,t)} - 1\right), & t \geq 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Problem 5. (25 points)

(a) (5 points)
Using the law of iterated expectations and the law of total variance,

$$\begin{aligned} \mathbf{E}[N] &= \mathbf{E}[\mathbf{E}[N \mid X]] \\ &= \mathbf{E}[X] \\ &= \frac{2}{\lambda}, \end{aligned}$$

$$\begin{aligned} \text{var}(N) &= \mathbf{E}[\text{var}(N \mid X)] + \text{var}(\mathbf{E}[N \mid X]) \\ &= \mathbf{E}[X] + \text{var}(X) \\ &= \frac{2}{\lambda} + \frac{2}{\lambda^2}, \end{aligned}$$

where $\text{var}(N \mid X) = \mathbf{E}[N \mid X] = X$.

(b) (5 points)

$$\begin{aligned} p_N(n) &= \int_x f_X(x) p_{N|X}(n \mid x) dx \\ &= \int_{x=0}^{\infty} \frac{\lambda^2}{n!} x^{n+1} e^{-(1+\lambda)x} dx \\ &= \frac{\lambda^2}{n!} \cdot \frac{(n+1)!}{(1+\lambda)^{n+2}} \\ &= \begin{cases} \frac{\lambda^2(n+1)}{(1+\lambda)^{n+2}} & n = 0, 1, 2 \ldots \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(c) (5 points)
The equation for $\hat{X}_{\text{lin}}(N)$, the linear least-squares estimator of X based on an observation of N, is

$$\hat{X}_{\text{lin}}(N) = \mathbf{E}[X] + \frac{\text{cov}(X, N)}{\text{var}(N)}(N - \mathbf{E}(N)).$$

The only unknown quantity is $\text{cov}(X, N) = \mathbf{E}[XN] - \mathbf{E}[X]\mathbf{E}[N] = \mathbf{E}[XN] - (\mathbf{E}[X])^2$. Using the law of iterated expectations again,

$$\begin{aligned}\mathbf{E}[XN] &= \mathbf{E}[\mathbf{E}[XN \mid X]] \\ &= \mathbf{E}[X\mathbf{E}[N \mid X]] \\ &= \mathbf{E}[X^2] = \text{var}(X) + (\mathbf{E}[X])^2 \\ &= \frac{6}{\lambda^2}.\end{aligned}$$

Thus, $\text{cov}(X, N) = 6/\lambda^2 - 4/\lambda^2 = 2/\lambda^2$. Combining this result with those from (a),

$$\begin{aligned}\hat{X}_{\text{lin}}(N) &= \frac{2}{\lambda} + \frac{\frac{2}{\lambda^2}}{\frac{2}{\lambda} + \frac{2}{\lambda^2}}\left(N - \frac{2}{\lambda}\right) \\ &= \frac{2+N}{1+\lambda}.\end{aligned}$$

(d) (5 points)
The expression for $\hat{X}_{\text{MAP}}(N)$, the MAP estimator of X based on an observation of N is

$$\begin{aligned}\hat{X}_{\text{MAP}}(N) &= \arg\max_x f_{X|N}(x \mid n) \\ &= \arg\max_x \frac{f_X(x)p_{N|X}(n \mid x)}{p_N(n)} \\ &= \arg\max_x f_X(x)p_{N|X}(n \mid x) \\ &= \arg\max_x \frac{\lambda^2}{n!}x^{n+1}e^{-(1+\lambda)x} \\ &= \arg\max_x x^{n+1}e^{-(1+\lambda)x},\end{aligned}$$

where the third equality holds since $p_N(n)$ has no dependency on x and the last equality holds by removing all quantities that have no dependency on x. The max can be found by differentiation and the result is:

$$\hat{X}_{\text{MAP}}(N) = \frac{1+N}{1+\lambda}.$$

This is the only local extremum in the range $x \in [0, \infty)$. Moreover, $f_{X|N}(x \mid n)$ equals 0 at $x = 0$ and goes to 0 as $x \to \infty$ and $f_{X|N}(x \mid n) > 0$ otherwise. We can therefore conclude that $\hat{X}_{\text{MAP}}(N)$ is indeed a maximum.

(e) (5 points)
To minimize the probability of error, we choose the hypothesis that has the larger posterior

probability. We will choose the hypothesis that $X = 2$ if

$$\begin{aligned}
\mathbf{P}(X = 2 \mid N = 3) &> \mathbf{P}(X = 3 \mid N = 3) \\
\frac{\mathbf{P}(X = 2)\mathbf{P}(N = 3 \mid X = 2)}{\mathbf{P}(N = 3)} &> \frac{\mathbf{P}(X = 3)\mathbf{P}(N = 3 \mid X = 3)}{\mathbf{P}(N = 3)} \\
\mathbf{P}(X = 2)\mathbf{P}(N = 3 \mid X = 2) &> \mathbf{P}(X = 3)\mathbf{P}(N = 3 \mid X = 3) \\
\frac{3^3}{35} \cdot \frac{2^3 e^{-2}}{3!} &> \frac{2^3}{35} \cdot \frac{3^3 e^{-3}}{3!} \\
e^{-2} &> e^{-3}.
\end{aligned}$$

The inequality holds so we choose the hypothesis that $X = 2$ to minimize the probability of error.

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