MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 14 Solutions
October 26, 2010

1. (a) Let X = (time between successive mosquito bites) = (time until the next mosquito bite).

   The mosquito bites occur according to a Bernoulli process with parameter $p = 0.5 \cdot 0.2 = 0.1$. X is a geometric random variable, so, $\mathbf{E}[X] = \frac{1}{p} = \frac{1}{0.1} = 10$.

$$\text{var}(X) = \frac{1-p}{p^2} = \frac{1-0.1}{0.1^2} = 90.$$

   (b) Mosquito bites occur according to a Bernoulli process with parameter $p = 0.1$. Tick bites occur according to another independent Bernoulli process with parameter $q = 0.1 \cdot 0.7 = 0.07$. Bug bites (mosquito or tick) occur according to a merged Bernoulli process from the mosquito and tick processes. Therefore, the probability of success at any time point for the merged Bernoulli process is $r = p + q - pq = 0.1 + 0.07 - 0.1 \cdot 0.07 = 0.163$. Let Y be the time between successive bug bites. As before, Y is a geometric random variable, so $\mathbf{E}[Y] = \frac{1}{r} = \frac{1}{0.163} \approx 6.135$.

$$\text{var}(Y) = \frac{1-r}{r^2} = \frac{1-0.163}{0.163^2} \approx 31.503$$

2. (a) In this case, since the trials are independent, the given information is irrelevant. $\mathbf{P}(\text{next 2 trials result in 3 tails}) = (\frac{1}{8})^2 = \frac{1}{64}$.

   (b) i. The second order Pascal PMF for random variable N, as defined in the text, is the probability of the second success comes on the n^{th} trial. Thus, the random variable, K, is a shifted version of the second order Pascal PMF, i.e. $K = N - 1$. So, the probability that 1 success comes in the first k trials, where the next trial will result in the second success, can be expressed as:

$$p_K(k) = \binom{k}{1}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{k-1}, \quad k \geq 1.$$

   ii. The number of tails before the first success, M, can be written as a random sum:

$$M = X_1 + X_2 + \cdots + X_N,$$

   where X_i is the number of tails that occur on (unsuccessful) trial i, and N is the number of unsuccessful trials (i.e. trials before the first success). We notice that X is equally likely to be either 1 or 2, and that N is a shifted geometric: $N = R - 1$, where R is a geometric random variable with parameter $\frac{1}{4}$. Now we can apply our random sum formulae.

$$E[M] = E[X]E[N] = (\frac{3}{2})(4-1) = \frac{9}{2}$$

$$\text{var}(M) = E[N]\text{var}(X) + (E[X])^2\text{var}(N) = (4-1)(\frac{1}{4}) + (\frac{3}{2})^2(12) = \frac{111}{4}.$$

(c) N, the number of trials in Bob's experiment, can be expressed as the sum of 3 independent random variables, X, Y, and Z. X is the number of trials until Bob removes the first coin, Y the number of additional trials until he removes the second coin, and Z the additional number until he removes the third coin. We see that X is a geometric random variable with parameter $\frac{1}{8}$, Y is geometric with parameter $\frac{1}{4}$, and Z geometric with parameter $\frac{1}{2}$. Hence,

$$E[N] = E[X] + E[Y] + E[Z] = 8 + 4 + 2 = 14.$$

3. Let M be the total number of draws you make until you have signed all n papers. Let T_i be the number of draws you make until drawing the next unsigned paper after having signed i papers. Then $M = T_0 + \cdots + T_{n-1}$.

   We can view the process of selecting the next unsigned paper after having signed i papers as a sequence of independent Bernoulli trials with probability of success $p_i = \frac{n-i}{n}$, since there are $n-i$ unsigned papers out of a total of n papers and receiving any paper is equally likely in a particular draw. The PMF governing the number of attempts we make until we succeed in drawing the next unsigned paper after having signed i papers is geometric. More concretely, the probability that it takes k tries to draw the next unsigned paper after having signed i papers is

$$\mathbf{P}(T_i = k) = (1 - p_i)^{k-1} p_i.$$

   With this model, the expected value of M, the number of draws you make until you sign all n papers is:

$$\mathbf{E}[M] = \mathbf{E}\left[\sum_{i=0}^{n-1} T_i\right] = \sum_{i=0}^{n-1} \mathbf{E}[T_i] = \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{k=1}^{n} \frac{1}{k}.$$

   For large n, this is on the order of: $n \int_1^n \frac{1}{x}\, dx = n \log n$.

6.041 / 6.431 Probabilistic Systems Analysis and Applied Probability
Fall 2010