MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

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1. (a) Let X_i be a random variable indicating the quality of the ith bulb ("1" for good bulbs, "0" for bad ones). X_i's are independent Bernoulli random variables. Let Z_n be

$$Z_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$

$$\mathbf{E}[Z_n] = p \qquad \text{var}(Z_n) = \frac{n\text{var}(X_i)}{n^2} = \frac{\sigma^2}{n},$$

where σ^2 is the variance of X_i.

Applying Chebyshev's inequality yields,

$$\mathbf{P}\left(|Z_n - p| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2},$$

As $n \to \infty$, $\frac{\sigma^2}{n\epsilon^2} \to 0$ and $\mathbf{P}\left(|Z_n - p| \geq \epsilon\right) \to 0$.

Hence, Z_n converges to p in probability.

(b) By Chebychev's inequality,

$$\mathbf{P}\left(|Z_{50} - p| \geq 0.1\right) \leq \frac{\sigma^2}{50(0.1)^2},$$

Since X_i is a Bernoulli random variable, its variance σ^2 is $p(1-p)$, which is less than or equal to $\frac{1}{4}$. Thus,

$$\mathbf{P}\left(|Z_{50} - p| \geq 0.1\right) \leq \frac{1/4}{50(0.1)^2} = 0.5$$

(c) By Chebychev's inequality,

$$\mathbf{P}\left(|Z_n - p| \geq 0.1\right) \leq \frac{\sigma^2}{n\epsilon^2} \leq \frac{1/4}{n(0.1^2)}$$

To guarantee a probability 0.95 of falling in the desired range,

$$\frac{1/4}{n(0.1)^2} < 0.05,$$

which yields $n \geq 500$. Note that $n \geq 500$ guarantees the accuracy specification even for the highest variance, namely 1/4. For smaller variances, we need smaller values of n to guarantee the desired accuracy. For example, if $\sigma^2 = 1/16$, $n \geq 125$ would suffice.

2. (a) $\mathbf{E}[X_n] = 0 \cdot \left(1 - \frac{1}{n}\right) + 1 \cdot \frac{1}{n} = \frac{1}{n}$

$\text{var}(X_n) = \left(0 - \frac{1}{n}\right)^2 \cdot \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)^2 \cdot \left(\frac{1}{n}\right) = \frac{n-1}{n^2}$

$\mathbf{E}[Y_n] = 0 \cdot \left(1 - \frac{1}{n}\right) + n \cdot \frac{1}{n} = 1$

$\text{var}(Y_n) = (0-1)^2 \cdot \left(1 - \frac{1}{n}\right) + (n-1)^2 \cdot \left(\frac{1}{n}\right) = n - 1$

(b) Using Chebyshev's inequality, we have
$\lim_{n\to\infty} \mathbf{P}(|X_n - \frac{1}{n}| \geq \epsilon) \leq \lim_{n\to\infty} \frac{n-1}{n^2\epsilon^2} = 0$
Moreover, $\lim_{n\to\infty} \frac{1}{n} = 0$.
It follows that X_n converges to 0 in probability. For Y_n, Chebyshev suggests that,

$$\lim_{n\to\infty} \mathbf{P}(|Y_n - 1| \geq \epsilon) \leq \lim_{n\to\infty} \frac{n-1}{\epsilon^2} = \infty,$$

Thus, we cannot conclude anything about the convergence of Y_n through Chebychev's inequality.

(c) For every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}(|Y_n| \geq \epsilon) \leq \lim_{n\to\infty} \frac{1}{n} = 0,$$

Thus, Y_n converges to zero in probability.

(d) The statement is false. A counter example is Y_n. It converges in probability to 0 yet its expected value is 1 for all n.

(e) Using the Markov bound, we have

$$\mathbf{P}\left(|X_n - c| \geq \epsilon\right) = P\left(|X_n - c|^2 \geq \epsilon^2\right) \leq \frac{\mathbf{E}\left[(X_n - c)^2\right]}{c^2}.$$

Taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mathbf{P}\left(|X_n - c| \geq \epsilon\right) = 0,$$

which establishes convergence in probability.

(f) A counter example is Y_n. Y_n converges to 0 in probability, but

$$\mathbf{E}\left[(Y_n - 0)^2\right] = 0 \cdot \left(1 - \frac{1}{n}\right) + (n^2).\frac{1}{n} = n$$

Thus,

$$\lim_{n\to\infty} \mathbf{E}\left[(Y_n - 0)^2\right] = \infty,$$

and Y_n does not converge to 0 in the mean square.

3. (a) No. Since X_i for any $i \geq 1$ is uniformly distributed between -1.0 and 1.0.

(b) Yes, to 0. Since for $\epsilon > 0$,

$$\begin{aligned}
\lim_{i\to\infty} \mathbf{P}\left(|Y_i - 0| > \epsilon\right) &= \lim_{i\to\infty} \mathbf{P}\left(|\frac{X_i}{i} - 0| > \epsilon\right) \\
&= \lim_{i\to\infty} \left[\mathbf{P}\left(X_i > i\epsilon\right) + \mathbf{P}\left(X_i < -i\epsilon\right)\right] = 0.
\end{aligned}$$

(c) Yes, to 0. Since for $\epsilon > 0$,

$$\begin{aligned}
\lim_{i\to\infty} \mathbf{P}\left(|Z_i - 0| > \epsilon\right) &= \lim_{i\to\infty} \mathbf{P}\left(|(X_i)^i - 0| > \epsilon\right) \\
&= \lim_{i\to\infty} \left[\mathbf{P}(X_i > \epsilon^{\frac{1}{i}}) + \mathbf{P}(X_i < -(\epsilon)^{\frac{1}{i}})\right] \\
&= \lim_{i\to\infty} \left[\frac{1}{2}(1 - \epsilon^{\frac{1}{i}}) + \frac{1}{2}(1 - \epsilon^{\frac{1}{i}})\right] = \lim_{i\to\infty} \left(1 - \sqrt[i]{\epsilon}\right) \\
&= 0.
\end{aligned}$$

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