LECTURE 14

The Poisson process

- **Readings:** Start Section 6.2.

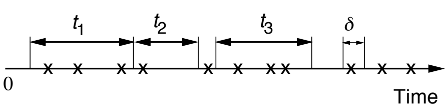
Lecture outline

- Review of Bernoulli process
- Definition of Poisson process
- Distribution of number of arrivals
- Distribution of interarrival times
- Other properties of the Poisson process

Bernoulli review

- Discrete time; success probability p
- Number of arrivals in n time slots: binomial pmf
- Interarrival times: geometric pmf
- Time to k arrivals: Pascal pmf
- Memorylessness

Definition of the Poisson process

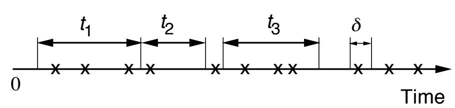


- **Time homogeneity:**
  $P(k, \tau)$ = Prob. of k arrivals in interval of duration τ
- Numbers of arrivals in disjoint time intervals are **independent**
- **Small interval probabilities:**
  For VERY small δ:

$$P(k, \delta) \approx \begin{cases} 1 - \lambda\delta, & \text{if } k = 0; \\ \lambda\delta, & \text{if } k = 1; \\ 0, & \text{if } k > 1. \end{cases}$$

  - λ: "arrival rate"

PMF of Number of Arrivals N



- Finely discretize $[0, t]$: approximately Bernoulli
- N_t (of discrete approximation): binomial
- Taking $\delta \to 0$ (or $n \to \infty$) gives:

$$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- $\mathbf{E}[N_t] = \lambda t, \qquad \text{var}(N_t) = \lambda t$

Example

- You get email according to a Poisson process at a rate of $\lambda = 5$ messages per hour. You check your email every thirty minutes.

- Prob(no new messages) =

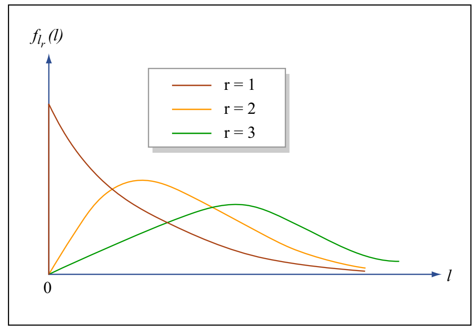
- Prob(one new message) =

Interarrival Times

- Y_k time of kth arrival

- **Erlang** distribution:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \geq 0$$



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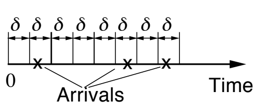
- Time of first arrival ($k = 1$): **exponential**: $f_{Y_1}(y) = \lambda e^{-\lambda y}, \quad y \geq 0$
  - Memoryless property: The time to the next arrival is independent of the past

Bernoulli/Poisson Relation



$n = t/\delta$ $\qquad np = \lambda t$

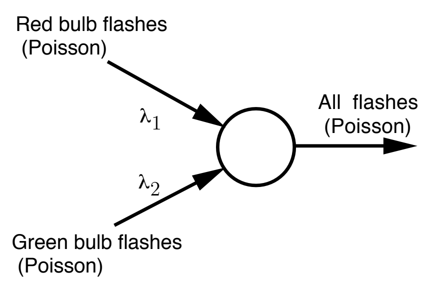
$p = \lambda\delta$

	POISSON	BERNOULLI
Times of Arrival	Continuous	Discrete
Arrival Rate	λ/unit time	p/per trial
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time Distr.	Exponential	Geometric
Time to k-th arrival	Erlang	Pascal

Merging Poisson Processes

- Sum of independent Poisson **random variables** is Poisson

- Merging of independent Poisson **processes** is Poisson



  - What is the probability that the next arrival comes from the first process?

6.041 / 6.431 Probabilistic Systems Analysis and Applied Probability
Fall 2010