Recitation 14
April 11, 2006

1. Suppose four random variables, W, X, Y and Z, are known to be pairwise uncorrelated and to satisfy
$$\mathbf{E}[W] = \mathbf{E}[X] = \mathbf{E}[Y] = \mathbf{E}[Z] = 0$$
and
$$\text{var}(W) = \text{var}(X) = \text{var}(Y) = \text{var}(Z) = 1.$$
Let $A = W + X$, $B = X + Y$ and $C = Y + Z$. Compute $\mathbf{E}[AB]$ and $\mathbf{E}[AC]$, or the correlation between A & B and A & C, respectively.

2. (Problem 4.25) **Correlation Coefficient**.
Consider the correlation coefficient
$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}$$
of two random variables X and Y that have positive variances. Show that:
   (a) $|\rho(X, Y)| \leq 1$. *Hint*: Use the Schwarz inequality:
$$(\mathbf{E}[XY])^2 \leq \mathbf{E}[X^2]\mathbf{E}[Y^2].$$
   (b) If $Y - \mathbf{E}[Y]$ is a positive multiple of $X - \mathbf{E}[X]$, then $\rho(X, Y) = 1$.
   (c) If $Y - \mathbf{E}[Y]$ is a negative multiple of $X - \mathbf{E}[X]$, then $\rho(X, Y) = -1$.
   (d) If $\rho(X, Y) = 1$, then, with probability 1, $Y - \mathbf{E}[Y]$ is a positive multiple of $X - \mathbf{E}[X]$.
   (e) If $\rho(X, Y) = -1$, then, with probability 1, $Y - \mathbf{E}[Y]$ is a negative multiple of $X - \mathbf{E}[X]$.

3. (Problem 4.29) Let X and Y be two random variables with positive variances.
   (a) Let $\hat{X}_L$ be the linear least mean squares estimator of X based on Y. Show that
$$\mathbf{E}[(X - \hat{X}_L)Y] = 0.$$
   Use this property to show that the correlation of the estimation error $X - \hat{X}_L$ with Y is zero.
   (b) Let $\hat{X} = \mathbf{E}[X \mid Y]$ be the least mean squares estimator of X given Y. Show that
$$\mathbf{E}[(X - \hat{X})h(Y)] = 0,$$
   for any function h.
   (c) Is it true that the estimation error $X - \mathbf{E}[X \mid Y]$ is independent of Y?