CHAPTER FIVE

discrete-state Markov processes

The Bernoulli and Poisson processes are defined by probabilistic descriptions of series of *independent* trials. The Markov process is one type of characterization of a series of *dependent* trials.

We have emphasized the no-memory properties of the Bernoulli and Poisson processes. Markov processes do have memory (events in nonoverlapping intervals of time need not be independent), but the dependence of future events on past events is of a particularly simple nature.

5-1 Series of Dependent Trials for Discrete-state Processes

Consider a system which may be described at any time as being in one of a set of mutually exclusive collectively exhaustive *states* $S_1, S_2, \ldots, S_m$. According to a set of probabilistic rules, the system may, at certain discrete instants of time, undergo *changes of state* (or *state transitions*). We number the particular instants of time at which transitions may occur, and we refer to these instants as the *first trial*, the *second trial*, etc.

Let $S_i(n)$ be the event that the system is in state S_i immediately after the nth trial. The probability of this event may be written as $P[S_i(n)]$. Each trial in the general process of the type (*discrete state, discrete transition*) introduced in the above paragraph may be described by transition probabilities of the form

$$P[S_j(n) \mid S_a(n-1)S_b(n-2)S_c(n-3) \cdots]$$
$$1 \le j, a, b, c, \ldots, \le m; \quad n = 1, 2, 3, \ldots$$

These transition probabilities specify the probabilities associated with each trial, and they are conditional on the entire past history of the process. The above quantity, for instance, is the conditional probability that the system will be in state S_j immediately after the nth trial, given that the previous state history of the process is specified by the event $S_a(n-1)S_b(n-2)S_c(n-3) \cdots$.

We note some examples of series of dependent trials in discrete-state discrete-transition processes. The states might be nonnegative integers representing the number of people on a bus, and each bus stop might be a probabilistic trial at which a change of state may occur. Another example is a process in which one of several biased coins is flipped for each trial and the selection of the coin for each trial depends in some manner on the outcomes of the previous flips. The number of items in a warehouse at the start of each day is one possible state description of an inventory. For this process, the state transition due to the total transactions on any day could be considered to be the result of one of a continuing series of dependent trials.

5-2 Discrete-state Discrete-transition Markov Processes

If the transition probabilities for a series of dependent trials satisfy the

Markov condition:

$$P[S_j(n) \mid S_a(n-1)S_b(n-2)S_c(n-3) \cdots] = P[S_j(n) \mid S_a(n-1)] \quad \text{for all } n, j, a, b, c, \ldots$$

the system is said to be a *discrete-state discrete-transition Markov process*.

If the state of the system immediately prior to the nth trial is known, the Markov condition requires that the conditional transition probabilities describing the nth trial do not depend on any additional past history of the process. The present state of the system specifies all historical information relevant to the future behavior of a Markov process.

We shall not consider processes for which the conditional transition probabilities

$$P[S_j(n) \mid S_i(n-1)]$$

depend on the number of the trial. Thus we may define the *state transition probabilities* for a discrete-transition Markov process to be

$$p_{ij} = P[S_j(n) \mid S_i(n-1)] \qquad 1 \le i, j \le m; \quad p_{ij} \text{ independent of } n$$

Quantity p_{ij} is the conditional probability that the system will be in state S_j immediately after the next trial, given that the present state of the process is S_i. We always have $0 \le p_{ij} \le 1$, and, because the list of states must be mutually exclusive and collectively exhaustive, it must also be true that

$$\sum_j p_{ij} = 1 \qquad \text{for } i = 1, 2, 3, \ldots, m$$

It is often convenient to display these transition probabilities as members of an $m \times m$ *transition matrix* $[p]$, for which p_{ij} is the entry in the ith row and jth column

$$[p] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

We also define the *k-step state transition probability* $p_{ij}(k)$,

$$p_{ij}(k) = \begin{pmatrix} \text{conditional probability that proc-} \\ \text{ess will be in state } S_j \text{ after exactly} \\ k \text{ more trials, given that present} \\ \text{state of process is } S_i \end{pmatrix} = P[S_j(n+k) \mid S_i(n)]$$

$$p_{ij}(0) = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases} \qquad p_{ij}(1) = p_{ij}$$

Consider any integer l, subject to $0 \le l \le k$. We may always write

$$p_{ij}(k) = P[S_j(n+k) \mid S_i(n)] = \sum_{x=1}^{m} P[S_j(n+k)S_x(n+k-l) \mid S_i(n)]$$

which simply notes that the process had to be in *some* state immediately after the $(n+k-l)$th trial. From the definition of conditional probability we have

$$P[S_j(n+k)S_x(n+k-l) \mid S_i(n)]$$
$$= P[S_x(n+k-l) \mid S_i(n)] \cdot P[S_j(n+k) \mid S_x(n+k-l)S_i(n)]$$

For a discrete-state discrete-transition Markov process we may use the Markov condition on the right-hand side of this equation to obtain

$$P[S_j(n+k) \mid S_x(n+k-l)S_i(n)] = P[S_j(n+k) \mid S_x(n+k-l)]$$
$$P[S_j(n+k)S_x(n+k-l) \mid S_i(n)] = p_{ix}(k-l)p_{xj}(l)$$

which may be substituted in the above equation for $p_{ij}(k)$ to obtain the result

$$p_{ij}(k) = \sum_{x=1}^{m} p_{ix}(k-l)p_{xj}(l)$$
$$k = 1, 2, 3, \ldots; \quad 0 \le l \le k; \quad 1 \le i, j \le m$$

This relation is a simple case of the *Chapman-Kolmogorov* equation, and it may be used as an alternative definition for the discrete-state discrete-transition Markov process with constant transition probabilities. This equation need *not* apply to the more general process described in Sec. 5-1.

Note that the above relation, with $l = 1$,

$$p_{ij}(k) = \sum_{x=1}^{m} p_{ix}(k-1)p_{xj}$$

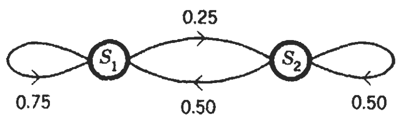
provides a means of calculation of the k-step transition probabilities which is more efficient than preparing a probability tree for k trials and then collecting the probabilities of the appropriate events (see Prob. 5.02).

We consider one example. Suppose that a series of dependent-coin flips can be described by a model which assigns to any trial conditional probabilities which depend only on the outcome of the previous trial. In particular, we are told that any flip immediately following an experimental outcome of a head has probability 0.75 of also resulting in a head and that any flip immediately following a tail is a fair toss. Using the most recent outcome as the state description, we have the two-state Markov process

S_1: heads

S_2: tails

$$[p] = \begin{bmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{bmatrix}$$



In the *state-transition diagram* shown above, we have made a picture of the process in which the states are circles and the trial transition probabilities are labeled on the appropriate arrowed branches.

We may use the relation

$$p_{ij}(k) = \sum_{x=1}^{m} p_{ix}(k-1)p_{xj}$$

first for $k = 2$, then for $k = 3$, etc., to compute the following table (in which we round off to three significant figures):

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$p_{11}(k)$	0.750	0.688	0.672	0.668	0.667	0.667	0.667	0.667
$p_{12}(k)$	0.250	0.312	0.328	0.332	0.333	0.333	0.333	0.333
$p_{21}(k)$	0.500	0.625	0.656	0.664	0.666	0.667	0.667	0.667
$p_{22}(k)$	0.500	0.375	0.344	0.336	0.334	0.333	0.333	0.333

Our table informs us, for instance, that, given the process is in state S_1 at any time, the conditional probability that the process will be in state S_2 exactly three trials later is equal to 0.328. In this example it appears that the k-step transition probabilities $p_{ij}(k)$ reach a limiting value as k increases and that these limiting values do not depend on i. We shall study this important property of *some* Markov processes in the next few sections.

If the probabilities describing each trial had depended on the results of the previous C flips, the resulting sequence of dependent trials could still be represented as a Markov process. However, the state description might require as many as 2^C states (for an example, see Prob. 5.01).

It need not be obvious whether or not a particular physical system can be modeled accurately by a Markov process with a finite number of states. Often this turns out to depend on how resourceful we are in suggesting an appropriate state description for the physical system.

5-3 State Classification and the Concept of Limiting-state Probabilities

We observed one interesting result from the dependent coin-flip example near the end of Sec. 5-2. As $k \to \infty$, the k-step state transition probabilities $p_{ij}(k)$ appear to depend neither on k nor on i.

If we let $P[S_i(0)]$ be the probability that the process is in state S_i just before the first trial, we may use the definition of $p_{ij}(k)$ to write

$$P[S_j(k)] = \sum_{i=1}^{m} P[S_i(0)]p_{ij}(k)$$

The quantities $P[S_i(0)]$ are known as the *initial conditions* for the process. If it is the case that, as $k \to \infty$, the quantity $p_{ij}(k)$ depends neither on k nor on i, then we would conclude from the above equation that $P[S_j(k)]$ approaches a constant as $k \to \infty$ and this constant is independent of the initial conditions.

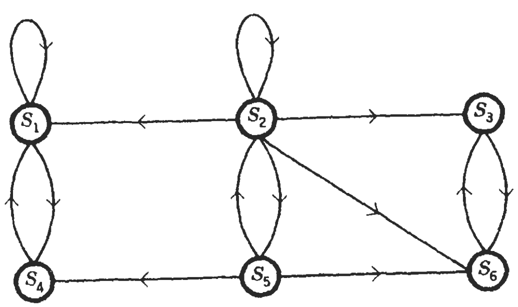
Many (*but not all*) Markov processes do in fact exhibit this behavior. For processes for which the *limiting-state probabilities*

$$\lim_{k \to \infty} P[S_j(k)] = P_j \qquad j = 1, 2, \ldots, m$$

exist and are independent of the initial conditions, many significant questions may be answered with remarkable ease. A correct discussion of this matter requires several definitions.

State S_i is called *transient* if there exists a state S_j and an integer l such that $p_{ij}(l) \neq 0$ and $p_{ji}(k) = 0$ for $k = 0, 1, 2, \ldots$. This simply states that S_i is a transient state if there exists any state to which the system (in some number of trials) can get to from S_i but from which it can never return to S_i. For a Markov process with a finite number of states, we might expect that, after very many trials, the probability that the process is in any transient state approaches zero, no matter what the initial state of the process may have been.

As an example, consider the process shown below,



for which we have indicated branches for all state transitions which are to have nonzero transition probabilities. States S_2 and S_5 are the only states which the process can leave in some manner such that it may never return to them; so S_2 and S_5 are the only transient states in this example.

State S_i is called *recurrent* if, for every state S_j, the existence of an integer r_j such that $p_{ij}(r_j) > 0$ implies the existence of an integer r_i such that $p_{ji}(r_i) > 0$. From this definition we note that, no matter what state history may occur, once the process enters a recurrent state it will *always* be possible, in some number of transitions, to return to that state. Every state must be either recurrent or transient. In the above example, states S_1, S_3, S_4, and S_6 are recurrent states.

The fact that each of two states is recurrent does not necessarily require that the process can ever get from one of these states to the other. One example of two recurrent states with $p_{ij}(k) = p_{ji}(k) = 0$

for all k is found by considering the pair of states S_1 and S_3 in the above diagram.

Recurrent state S_i is called *periodic* if there exists an integer d, with $d > 1$, such that $p_{ii}(k)$ is equal to zero for all values of k other than $d, 2d, 3d, \ldots$. In our example above, recurrent states S_3 and S_6 are the only periodic states. (For our purposes, there is no reason to be concerned with periodicity for transient states.)

A set W of recurrent states forms one *class* (or a *single chain*) if, for every pair of recurrent states S_i and S_j of W, there exists an integer r_{ij} such that $p_{ij}(r_{ij}) > 0$. Each such set W includes all its possible members. The members of a class of recurrent states satisfy the condition that it is possible for the system (eventually) to get from any member state of the class to any other member state. In our example, there are two single chains of recurrent states. One chain is composed of states S_1 and S_4, and the other chain includes S_3 and S_6. Note that the definition of a single chain is concerned only with the properties of the recurrent states of a Markov process.

After informally restating these four definitions for m-state Markov processes ($m < \infty$), we indicate why they are of interest.

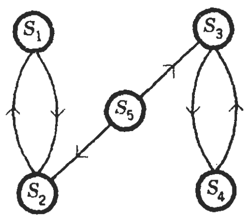
TRANSIENT STATE S_i: From at least one state which may be reached eventually from S_i, system can never return to S_i.

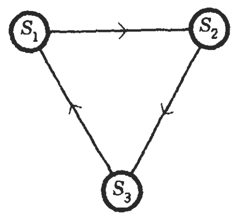
RECURRENT STATE S_i: From every state which may be reached eventually from S_i, system can eventually return to S_i.

PERIODIC STATE S_i: A recurrent state for which $p_{ii}(k)$ may be nonzero only for $k = d, 2d, 3d, \ldots$, with d an integer greater than unity.

SINGLE CHAIN W: A set of recurrent states with the property that the system can eventually get from any member state to any other state which is also a member of the chain. All possible members of each such chain are included in the chain.

For a Markov process with a finite number of states *whose recurrent states form a single chain* and *which contains no periodic states*, we might expect that the k-step transition probabilities $p_{ij}(k)$ become independent of i and k as k approaches infinity. We might argue that such a process has "limited memory." Although successive trials are strongly dependent, it is hard to see how $P[S_i(k)]$ should be strongly influenced by either k or the initial state after a large number of trials. In any case, it should be clear that, for either of the following processes,





we would certainly *not* expect *any* $p_{ij}(k)$ to become independent of i and k as k gets very large.

We speculated that, for a Markov process with a finite number of states, whose recurrent states form a single chain, and which contains no periodic states, we might expect that

$$\lim_{k\to\infty} p_{ij}(k) = P_j \qquad \sum_{j=1}^{m} P_j = 1$$

where P_j depends neither on k nor on i. In fact this result is established by a simplified form of the *ergodic theorem*, which we shall state without proof in the following section. The P_j's, known as the *limiting-state probabilities*, represent the probabilities that a single-chain process with no periodic states will be in state S_j after very many trials, no matter what the initial conditions may have been.

Since our example of the dependent coin flips in the previous section satisfies these restrictions, the ergodic theorem states, for example, that quantity $P[S_1(n)]$ = Prob(heads on nth toss) will approach a constant as $n \to \infty$ and that this constant will not depend on the initial state of the process.

5-4 The Ergodic Theorem

In this section we shall present and discuss a formal statement of a simple form of the ergodic theorem for a discrete-state discrete-transition Markov process. The ergodic theorem is as follows:

Let M_k be the matrix of k-step transition probabilities of a Markov process with a finite number of states $S_1, S_2, \ldots, S_m$. If there exists an integer k such that the terms $p_{ij}(k)$ of the matrix M_k satisfy the relation

$$\min_{1\le i\le m} p_{ij}(k) = \delta > 0$$

for at least one column of M_k, then the equalities

$$\lim_{n\to\infty} p_{ij}(n) = P_j \qquad j = 1, 2, \ldots, m \qquad i = 1, 2, \ldots, m; \qquad \sum_j P_j = 1$$

are satisfied.

The restriction

$$\min_{1\le i\le m} p_{ij}(k) = \delta > 0$$

for at least one column of M_k simply requires that there be at least one state S_j and some number k such that it be possible to get to S_j from every state in exactly k transitions. This requirement happens to correspond to the conditions that the recurrent states of the system form a single chain and that there be no periodic states.

When the above restriction on the $p_{ij}(k)$ is satisfied for some value of k, the ergodic theorem states that, as $n \to \infty$, the n-step transition probabilities $p_{ij}(n)$ approach the limiting, or "steady-state," probabilities P_j. A formal test of whether this restriction does in fact hold for a given process requires certain matrix operations not appropriate to the mathematical background assumed for our discussions. We shall work with the "single chain, finite number of states, and no periodic states" restriction as being equivalent to the restriction in the ergodic theorem. (The single-chain and no-periodic-states restrictions are necessary conditions for the ergodic theorem; the finite-number-of-states restriction is not a necessary condition.) For the representative Markov systems to be considered in this book, we may test for these properties by direct observation.

5-5 The Analysis of Discrete-state Discrete-transition Markov Processes

We begin this section with a review of some of the things we already know about discrete-state discrete-transition Markov processes. We then write the general difference equations which describe the behavior of the state probabilities, the $P[S_j(n)]$'s, as the process operates over a number of trials. For processes to which the ergodic theorem applies, we also consider the solution of these difference equations as $n \to \infty$ to obtain the limiting-state probabilities. Finally, we note how our results simplify for the important class of Markov processes known as *birth-and-death* processes.

As we did at the beginning of our study of the Poisson process in Chap. 4, let us make use of an efficient but somewhat improper notation to suit our purposes. We define

$P_j(n) = P[S_j(n)]$ = probability process is in state S_j immediately after nth trial

From the definition of $p_{ij}(n)$ we may write

$$P_j(n) = \sum_i P_i(0)p_{ij}(n) \qquad \sum_j P_j(n) = 1 \qquad \text{for } n = 0, 1, 2, \ldots$$

where the $P_i(0)$'s, the *initial conditions*, represent the probabilities of the process being in its various states prior to the first trial. Because of the Markov condition, we also have

$$p_{ij}(n) = \sum_k p_{ik}(l)p_{kj}(n-l) \qquad l = 0, 1, 2, \ldots, n$$

and when the ergodic theorem applies, we know that

$$\lim_{n\to\infty} P_j(n) = P_j \qquad j = 1, 2, \ldots, m$$

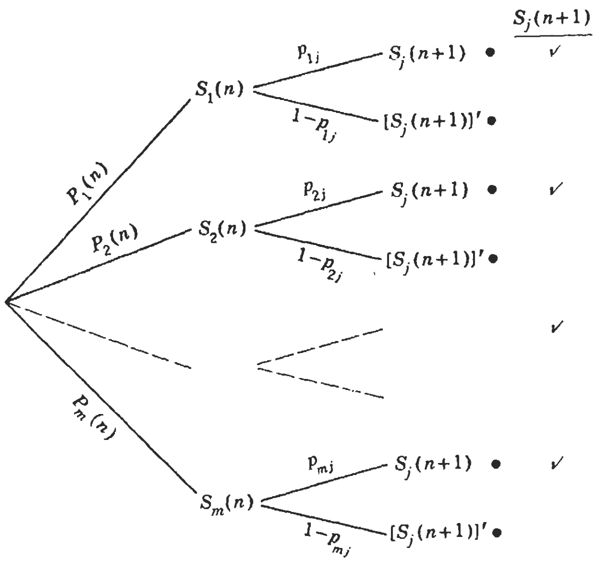
and these limits do not depend on the initial conditions.

One expresses his "state of knowledge" about the state of a Markov process at any time by specifying the *probability state vector*

$$\lfloor P(n) \rfloor = \lfloor P_1(n) \quad P_2(n) \quad \cdots \quad P_m(n) \rfloor$$

Now we shall write the describing equations for the state probabilities of a Markov process as it continues to undergo trials. There are efficient transform and flow-graph techniques for the general solution of these equations. Here we shall develop only the describing equations and consider their solution as n goes to infinity. Even this limited tour will prepare us to study many important practical systems.

The equations of interest will relate the $P_j(n+1)$'s to the state probabilities one trial earlier, the $P_i(n)$'s. Each of the first $m-1$ following relations sums the probabilities of the mutually exclusive and collectively exhaustive ways the event $S_j(n+1)$ can occur, in terms of the state probabilities immediately after the nth trial. We start with the appropriate event space,



We collect the probabilities of the events $S_j(n+1)$ for $j = 1$ (first equation), for $j = 2$ (second equation), etc., up through $j = m - 1$. Thus, for any discrete-state discrete-transition Markov process

$$m-1 \text{ eqs.:} \begin{cases} P_1(n+1) = P_1(n)p_{11} + P_2(n)p_{21} + \cdots + P_m(n)p_{m1} \\ P_2(n+1) = P_1(n)p_{12} + P_2(n)p_{22} + \cdots + P_m(n)p_{m2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_{m-1}(n+1) = P_1(n)p_{1(m-1)} + P_2(n)p_{2(m-1)} + \cdots + P_m(n)p_{m(m-1)} \end{cases}$$

$$m\text{th eq.:} \qquad 1 = P_1(n+1) + P_2(n+1) + \cdots + P_m(n+1)$$

where the mth equation states the dependence of the $P_j(n+1)$'s. Given a set of initial conditions $P_1(0)$, $P_2(0)$, $\ldots$, $P_m(0)$, these equations may be solved iteratively for $n = 0, 1, 2$, etc., to determine the state probabilities as a function of n. More advanced methods which allow one to obtain the probability state vector $\lfloor P(n) \rfloor$ in closed form will not be considered here.

For the remainder of this section, we consider only processes *with no periodic states* and *whose recurrent states form a single chain*. The ergodic theorem applies to such processes; so we may let

$$\lim_{n\to\infty} P_i(n+1) = \lim_{n\to\infty} P_i(n) = P_i$$

and, rewriting the above equations for the limit as $n \to \infty$, we have

$$m-1 \text{ eqs.:} \begin{cases} 0 = P_1(p_{11}-1) + P_2p_{21} + \cdots + P_mp_{m1} \\ 0 = P_1p_{12} + P_2(p_{22}-1) + \cdots + P_mp_{m2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = P_1p_{1(m-1)} + P_2p_{2(m-1)} + \cdots + P_mp_{m(m-1)} \end{cases}$$

$$m\text{th eq.:} \qquad 1 = P_1 + P_2 + \cdots + P_m$$

The solution to these m simultaneous equations determines the limiting-state probabilities for those processes which meet the restrictions for the ergodic theorem. Examples of the use of our two sets of simultaneous equations are given in the following section. (By writing the mth equation in the form used above, we avoid certain problems with this set of simultaneous equations.)

Happily, there is a special and very practical type of Markov process for which the equations for the limiting-state probabilities may be solved on sight. We further limit our discussion of single-chain processes with no periodicities to the case of *birth-and-death* processes.

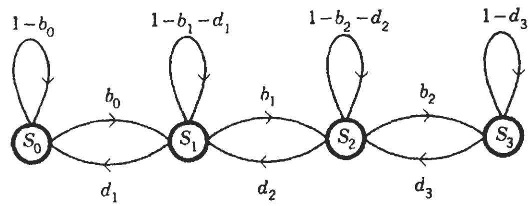
A discrete-state discrete-transition birth-and-death process is a Markov process whose transition probabilities obey

$$p_{ij} = 0 \qquad \text{if } j \neq i-1, \quad i, \quad i+1$$

and for these processes it is advantageous to adopt the *b*irth-and-*d*eath notation

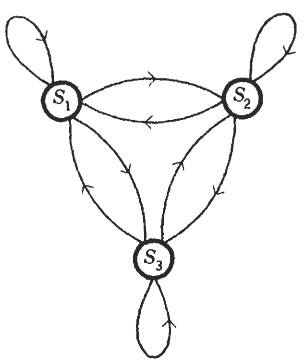
$$p_{i(i+1)} = b_i \qquad p_{i(i-1)} = d_i$$

One example of a birth-and-death process is



$$[p] = \begin{bmatrix} 1-b_0 & b_0 & 0 & 0 \\ d_1 & 1-b_1-d_1 & b_1 & 0 \\ 0 & d_2 & 1-b_2-d_2 & b_2 \\ 0 & 0 & d_3 & 1-d_3 \end{bmatrix}$$

Many practical instances of this type of process are mentioned in the problems at the end of this chapter. Note that our definition of the birth-and-death process (and the method of solution for the limiting-state probabilities to follow) does not include the process pictured below:



For the given assignment of state labels, this process will violate the definition of the birth and death process if either p_{13} or p_{31} is nonzero

We shall now demonstrate one argument for obtaining the

limiting-state probabilities for a single-chain birth-and-death process. We begin by choosing any *particular* state S_K and noting that, at any time in the history of a birth-and-death process, the total number of $S_K \rightarrow S_{K+1}$ transitions made so far must either be one less than, equal to, or one greater than the total number of $S_{K+1} \rightarrow S_K$ transitions made so far. (Try to trace out a possible state history which violates this rule.)

Consider the experiment which results if we approach a birth-and-death process after it has undergone a great many transitions and our state of knowledge about the process is given by the limiting-state probabilities. The probability that the *first* trial after we arrive will result in an $S_K \rightarrow S_{K+1}$ transition is $P_K b_K$; the probability that it will result in an $S_{K+1} \rightarrow S_K$ transition is $P_{K+1} d_{K+1}$.

Since, over a long period of time, the fractions of all trials which have these two outcomes must be equal and we are simply picking a trial at random, we must have (for a single-chain birth-and-death process with no periodic states)

$$P_K b_K = P_{K+1} d_{K+1}$$

and thus the limiting-state probabilities may be obtained by finding all P_i's in terms of P_0 from

$$P_{i+1} = \frac{P_i b_i}{d_{i+1}} \qquad i = 0, 1, 2, \ldots$$

and then solving for P_0 by using $\sum_i P_i = 1$

Another way to derive this result would be to notice that, for a birth-and-death process, many of the coefficients in the simultaneous equations for the P_i's for the more general single-chain Markov process are equal to zero. The resulting equations may easily be solved by direct substitution to obtain the solution stated above.

The first paragraph of this section may now serve as a road map for the above work. Several examples are discussed and solved in the following section.

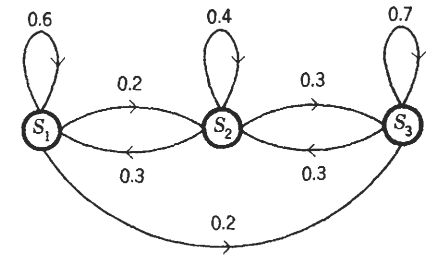
5-6 Examples Involving Discrete-transition Markov Processes

example 1 Experience has shown that the general mood of Herman may be realistically modeled as a three-state Markov process with the mutually exclusive collectively exhaustive states

S_1: Cheerful $\qquad S_2$: So-so $\qquad S_3$: Glum

His mood can change only overnight, and the following transition probabilities apply to each night's trial:

$$[p] = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix}$$



We are told that Herman's mood today is so-so.

(a) Determine the components of the probability state vector, the $P_i(n)$'s, for Herman's mood for the next few days.

(b) Determine this probability state vector for a day a few months hence. Is the answer dependent on the initial conditions?

(c) Determine the PMF for the number of trials until Herman's mood undergoes its first change of state.

(d) What is the probability that Herman will become glum before he becomes cheerful?

a We are given $\lfloor P(0) \rfloor = \lfloor P_1(0) \quad P_2(0) \quad P_3(0) \rfloor = \lfloor 0 \quad 1 \quad 0 \rfloor$, and we may use the original set of difference equations for the $P_j(n+1)$'s,

$$P_j(n+1) = \sum_i P_i(n)p_{ij} \qquad \text{for } j = 1, 2, \ldots, m-1$$

$$\sum_j P_j(n+1) = 1$$

first with $n = 0$, then with $n = 1$, etc. For instance, with $n = 0$ we find

$$P_1(1) = \sum_i P_i(0)p_{i1} = (0)(0.6) + (1)(0.3) + (0)(0.0) = 0.3$$

$$P_2(1) = \sum_i P_i(0)p_{i2} = (0)(0.2) + (1)(0.4) + (0)(0.3) = 0.4$$

$$1 = \sum_j P_j(1) = 0.3 + 0.4 + P_3(1) \qquad \therefore P_3(1) = 0.3$$

And thus we have obtained

$$\lfloor P(1) \rfloor = \lfloor P_1(1) \quad P_2(1) \quad P_3(1) \rfloor = \lfloor 0.3 \quad 0.4 \quad 0.3 \rfloor$$

Further iterations using the difference equations allow us to generate the following table:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$P_1(n)$	0.000	0.300	0.300	0.273	0.254	0.243	0.237
$P_2(n)$	1.000	0.400	0.310	0.301	0.303	0.305	0.306
$P_3(n)$	0.000	0.300	0.390	0.426	0.443	0.452	0.457

All entries have been rounded off to three significant figures. The difference equations apply to *any* discrete-state discrete-transition Markov process.

b Since our Markov model for this process has no periodic states and its recurrent states form a single chain, the limiting-state probabilities are independent of the initial conditions. (The limiting-state probabilities *would* depend on the initial conditions if, for instance, we had $p_{12} = p_{32} = p_{13} = p_{31} = 0$.) We shall assume that the limiting-state probabilities are excellent approximations to what we would get by carrying out the above table for about 60 more trials (two months). Thus we wish to solve the simultaneous equations for the limiting-state probabilities,

$$0 = \sum_i P_i p_{ij} - P_j \qquad j = 1, 2, \ldots, m-1$$

$$1 = \sum_j P_j$$

which, for our example, are

$$0 = P_1(0.6 - 1.0) + P_2(0.3) + P_3(0.0)$$

$$0 = P_1(0.2) + P_2(0.4 - 1.0) + P_3(0.3)$$

$$1 = P_1 + P_2 + P_3$$

which may be solved to obtain

$$P_1 = 3/13 \approx 0.231 \qquad P_2 = 4/13 \approx 0.308 \qquad P_3 = 6/13 \approx 0.461$$

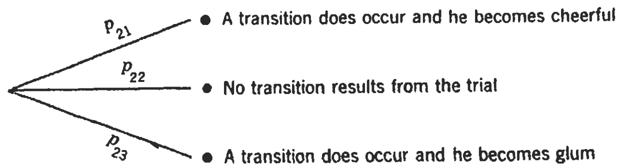
These values seem consistent with the behavior displayed in the above table. The probability that Herman will be in a glum mood 60 days hence is very close to 6/13. In fact, for this example, the limiting-state probabilities are excellent approximations to the actual-state probabilities 10 or so days hence. Note also that this is *not* a birth-and-death process ($p_{13} \neq 0$) and, therefore, we may not use the more rapid method of solution for the P_j's which applies only to birth-and-death processes.

c Given that Herman is still in state S_2, the conditional probability that he will undergo a change of state (of mind) at the next transition is given by $1 - p_{22}$. Thus the PMF for l, the number of (Bernoulli) trials up to and including his first change of mood, is the geometric PMF with parameter P equal to $1 - p_{22}$.

$$p_l(l_0) = (1 - p_{22})p_{22}^{l_0 - 1} = (0.6)(0.4)^{l_0 - 1} \qquad l_0 = 1, 2, 3, \ldots$$

We would obtain a similar result for the conditional PMF for the number of trials up to and including the next actual change of state for any discrete-transition Markov process, given the present state of the process. For this reason, one may say that such a process is characterized by *geometric holding times*. A similar phenomenon will be discussed in the introduction to the next section.

d Given that Herman's mood is so-so, the following event space describes any trial while he is in this state:

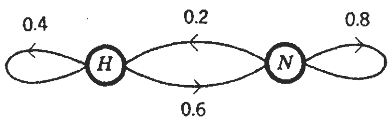


Thus we may calculate that the conditional probability he becomes glum given that a transition does occur is equal to $p_{23}/(p_{21} + p_{23})$. This is, of course, equal to the probability that he becomes glum before he becomes cheerful, and its numerical value is $0.3/(0.3 + 0.3) = 0.5$.

example 2 Roger Yogi Mantle, an exceptional baseball player who tends to have streaks, hit a home run during the first game of this season. The conditional probability that he will hit at least one homer in a game is 0.4 if he hit at least one homer in the previous game, but it is only 0.2 if he didn't hit any homers in the previous game. We assume this is a complete statement of the dependence. Numerical answers are to be correct within $\pm 2\%$.

(a) What is the probability that Roger hit at least one home run during the third game of this season?
(b) If we are told that he hit a homer during the third game, what is the probability that he hit at least one during the second game?
(c) If we are told that he hit a homer during the ninth game, what is the probability that he hit at least one during the tenth game?
(d) What is the probability that he will get at least one home run during the 150th game of this season?
(e) What is the probability that he will get home runs during both the 150th and 151st games of this season?
(f) What is the probability that he will get home runs during both the 3d and 150th games of this season?
(g) What is the probability that he will get home runs during both the 75th and 150th games of this season?

This situation may be formulated as a two-state Markov process. A game is type H if Roger hits at least one homer during the game; otherwise it is type N. For our model, we shall consider the trials to occur between games.

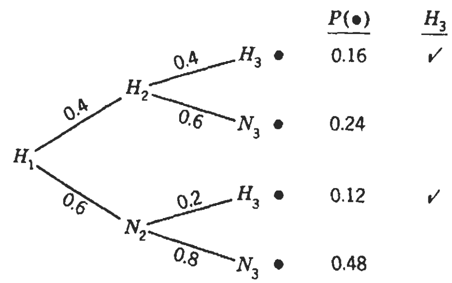


For this example we shall not go to an S_i description for each state, but we shall work directly with H and N, using the notation

$$P(H_n) = \text{probability Roger is in state } H \text{ during } n\text{th game}$$
$$P(N_n) = \text{probability Roger is in state } N \text{ during } n\text{th game}$$

We are given the initial condition $P(H_1) = 1$. We also note that this is a single-chain process with no periodic states, and it also happens to be a birth-and-death process.

a We wish to determine $P(H_3)$. One method would be to use the sequential sample space



to find $P(H_3) = 0.16 + 0.12 = 0.28$. Since the conditional branch traversal probabilities for the tree of a Markov process depend only on the most recent node, it is usually more efficient to solve for such state probabilities as a function of n from the difference equations, which, for this example, are

$$\left.\begin{aligned} P(H_{n+1}) &= 0.4P(H_n) + 0.2P(N_n) \\ 1 &= P(H_{n+1}) + P(N_{n+1}) \end{aligned}\right\} \qquad n = 1, 2, \ldots$$

and which lead, of course, to the same result.

b The desired conditional probability is easily calculated from the above sequential sample space,

$$P(H_2 \mid H_3) = \frac{P(H_2H_3)}{P(H_3)} = \frac{0.16}{0.28} = \frac{4}{7}$$

We have chosen to write $P(H_2 \mid H_3)$ rather than $P(H_2 \mid H_3H_1)$ because the event H_1 is given as part of the overall problem statement.

c The conditional probability that Roger hits at least one homer in the 10th game, given he hit at least one in the 9th game (and given no information about later games), is simply p_{HH}, which is given to be 0.4 in the problem statement.

d If we carry out a few iterations using the difference equations given after the solution to part (a) we find, working to three significant figures,

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$P(H_n)$	1.000	0.400	0.280	0.256	0.251	0.250	0.250	0.250
$P(N_n)$	0.000	0.600	0.720	0.744	0.749	0.750	0.750	0.750

Thus it is conservative to state that, for all practical purposes, his performances in games separated by more than 10 games may be considered independent events. $P(H_{150})$ is just the limiting-state probability P_H, which, taking advantage of our method for birth-and-death processes, is determined by

$$0.6P_H = 0.2P_N \qquad P_H + P_N = 1.0$$

resulting in $P_H = 0.25$, which checks with the result obtained by iteration above.

e The desired quantity is simply $P_H p_{HH} = \frac{1}{4} \cdot \frac{4}{10} = 0.1$. Note that the strong dependence of results in successive or nearly successive games must always be considered and that the required answer is certainly *not* P_H^2.

f $P(H_3H_{150}) = P(H_3)P(H_{150} \mid H_3) \approx P(H_3)P_H = (0.28)(\frac{1}{4}) = 0.07$

g Roger's performances on games this far apart may be considered independent events, and we have

$$P(H_{75}H_{150}) = P(H_{75})P(H_{150} \mid H_{75}) \approx P_H^2 = 1/16$$

The reader is reminded that these have been two elementary problems, intended to further our understanding of results obtained earlier in this chapter. Some more challenging examples will be found at the end of the chapter.

Some questions concerned with random incidence (Sec. 4-11) for a Markov process, a situation which did not arise here, will be introduced in the examples in Sec. 5-8.

5-7 Discrete-state Continuous-transition Markov Processes

Again we are concerned with a system which may be described at any time as being in one of a set of mutually exclusive collectively exhaustive discrete states $S_1, S_2, S_3, \ldots, S_m$. For a *continuous-transition* process, the probabilistic rules which describe the transition behavior allow changes of state to occur at any instants on a continuous time axis. If an observer knows the present state of *any* Markov process, any other information about the past state history of the process is irrelevant to his probabilistic description of the future behavior of the process.

In this section, we consider Markov processes for which, given that the present state is S_i, the conditional probability that an $S_i \to S_j$ transition will occur in the next Δt is given by $\lambda_{ij}\,\Delta t$ (for $j \neq i$ and suitably small Δt). Thus, each incremental Δt represents a trial whose outcome may result in a change of state, and the transition probabilities which describe these trials depend only on the present state of the system. *We shall not allow λ_{ij} to be a function of time;* this restriction corresponds to our not allowing p_{ij} to depend on the number of the trial in our discussion of the discrete-transition Markov process.

We begin our study of these discrete-state continuous-transition Markov processes by noting some consequences of the above description of the state transition behavior and by making some comparisons with discrete-transition processes. (All the following statements need hold only for suitably small Δt.)

The conditional probability that *no* change of state will occur in the next Δt, given that the process is at present in state S_i, is

$$\text{Prob(no change of state in next } \Delta t\text{, given present state is } S_i) = 1 - \sum_{\substack{j \\ j \neq i}} \lambda_{ij}\,\Delta t$$

Although p_{ii} was a meaningful parameter for the discrete-transition process, a quantity λ_{ii} has no similar interpretation in the continuous-transition process. This is one reason why our equations for the state probabilities as a function of time will be somewhat different in form from those describing the state probabilities for the discrete-transition process. (For reasons outside the scope of this text, it is preferable that we let λ_{ii} remain undefined rather than define λ_{ii} to be equal to zero.)

Given that the system is at present in state S_i, the probability of leaving this state in the next Δt, no matter how long the system has already been in state S_i, is equal to

$$\sum_{\substack{j \\ j \neq i}} \lambda_{ij}\,\Delta t$$

and, from our earlier study of the Poisson process, we realize that the remaining time until the next departure from the present state is an exponentially distributed random variable with expected value

$$\Big(\sum_{\substack{j \\ j \neq i}} \lambda_{ij}\Big)^{-1}$$

For this reason, the type of continuous process we have described is said to have *exponential holding times.* Surprisingly general physical systems, many of which do not have exponential holding times, may be modeled realistically by the resourceful use of such a Markov model.

For the continuous-transition process, we shall again define a transient state S_i to be one from which it is possible for the process eventually to get to some other state from which it can never return to S_i. A recurrent state S_i is one to which the system can return from any state which can eventually be reached from S_i. No concept of periodicity is required, and a single class (or chain) of recurrent states again includes all its possible members and has the property that it is possible eventually to get from any state which is a member of the class to any other member state.

A useful compact notation, similar to that used in Sec. 5-5, is $P_j(t) = P[S_j(t)] =$ probability process is in state S_j at time t. $P_j(t)$ must have the properties

$$0 \le P_j(t) \le 1 \qquad \sum_j P_j(t) = 1$$

We would expect, at least for a process with a finite number of states, that the probability of the process being in a transient state goes to zero as $t \to \infty$. For a recurrent state S_i in a single chain with a finite number of states we might expect

$$\int_0^\infty P_i(t)\, dt = \infty \qquad \text{if } S_i \text{ is a recurrent state in a single-chain process with a finite number of states}$$

since we expect $P_i(t)$ to approach a nonzero limit as $t \to \infty$.

We shall now develop the equations which describe the behavior of the state probabilities, the $P_i(t)$'s, as the process operates over time for any m-state continuous-transition Markov process with exponential holding times. The formulation is very similar to that used earlier for discrete-transition Markov processes. We shall write $m - 1$ incremental relations relating $P_j(t + \Delta t)$ to the $P_i(t)$'s, for $j = 1, 2, \ldots, m - 1$. Our mth equation will be the constraint that $\sum_j P_j(t + \Delta t) = 1$.

To express each $P_j(t + \Delta t)$, we sum the probabilities of all the mutually exclusive ways that the process could come to be in state S_j at $t + \Delta t$, in terms of the state probabilities at time t,

$m - 1$ eqs.:

$$\begin{cases} P_1(t + \Delta t) = P_1(t)\left(1 - \sum_{\substack{j \\ j\ne 1}} \lambda_{1j}\, \Delta t\right) + \sum_{\substack{j \\ j\ne 1}} P_j(t)\lambda_{j1}\, \Delta t \\ P_2(t + \Delta t) = P_2(t)\left(1 - \sum_{\substack{j \\ j\ne 2}} \lambda_{2j}\, \Delta t\right) + \sum_{\substack{j \\ j\ne 2}} P_j(t)\lambda_{j2}\, \Delta t \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_{m-1}(t + \Delta t) = P_{m-1}(t)\left(1 - \sum_{\substack{j \\ j\ne m-1}} \lambda_{(m-1)j}\, \Delta t\right) + \sum_{\substack{j \\ j\ne m-1}} P_j(t)\lambda_{j(m-1)}\, \Delta t \end{cases}$$

mth eq.: $\quad 1 = \sum_j P_j(t + \Delta t)$

On the right-hand side of the ith of the first $m - 1$ equations, the first term is the probability of the process being in state S_i at time t and not undergoing a change of state in the next Δt. The second term is the probability that the process entered state S_i from some other state during the incremental interval between t and $t + \Delta t$. We can simplify these equations by multiplying, collecting terms, dividing through by Δt, and taking the limit as $\Delta t \to 0$ to obtain, for *any* discrete-state continuous-transition Markov process with exponential holding times

$m - 1$ eqs.:

$$\begin{cases} \dfrac{dP_1(t)}{dt} = \sum_{\substack{j \\ j\ne 1}} P_j(t)\lambda_{j1} - P_1(t) \sum_{\substack{j \\ j\ne 1}} \lambda_{1j} \\ \dfrac{dP_2(t)}{dt} = \sum_{\substack{j \\ j\ne 2}} P_j(t)\lambda_{j2} - P_2(t) \sum_{\substack{j \\ j\ne 2}} \lambda_{2j} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{dP_{m-1}(t)}{dt} = \sum_{\substack{j \\ j\ne m-1}} P_j(t)\lambda_{j(m-1)} - P_{(m-1)}(t) \sum_{\substack{j \\ j\ne m-1}} \lambda_{(m-1)j} \end{cases}$$

mth eq.: $\quad 1 = \sum_j P_j(t)$

Each of the first $m - 1$ equations above relates the rate of change of a state probability to the probability of being elsewhere (in the first term) and to the probability of being in that state (in the second term). The solution of the above set of simultaneous differential equations, subject to a given set of initial conditions, would provide the state probabilities, the $P_i(t)$'s, for $i = 1, 2, \ldots, m$ and $t \ge 0$. Effective flow-graph and transform techniques for the solution of these equations exist but are outside the scope of our discussion. For some simple cases, as in Example 1 in Sec. 5-8, the direct solution of these equations presents no difficulties.

For the remainder of this section we limit our discussion to *processes whose recurrent states form a single chain.* We might expect for such processes that the effects of the initial conditions vanish as $t \to \infty$ and that $P_i(t + \Delta t) \to P_i(t)$ $\left(\text{or } \dfrac{dP_i(t)}{dt} \to 0\right)$ as $t \to \infty$. We define the limiting-state (or steady-state) probabilities by

$$\lim_{t\to\infty} P_i(t) = P_i$$

And we comment without proof that a suitable ergodic theorem does exist to establish the validity of the above speculations.

To obtain the equations for the limiting-state probabilities, we need only rewrite the simultaneous differential equations for the limiting case of $t \to \infty$. There results

$$0 = \sum_{\substack{j \\ j \neq 1}} P_j \lambda_{j1} - P_1 \sum_{\substack{j \\ j \neq 1}} \lambda_{1j}$$

$$0 = \sum_{\substack{j \\ j \neq 2}} P_j \lambda_{j2} - P_2 \sum_{\substack{j \\ j \neq 2}} \lambda_{2j}$$

. . . .

$$0 = \sum_{\substack{j \\ j \neq m-1}} P_j \lambda_{j(m-1)} - P_{m-1} \sum_{\substack{j \\ j \neq m-1}} \lambda_{(m-1)j}$$

$$1 = \sum_j P_j$$

If the process has been in operation for a long time, this term, multiplied by Δt, is the probability that the process will enter S_2 from elsewhere in a randomly selected Δt.

If the process has been in operation for a long time, this term, multiplied by Δt, is the probability that, in a randomly selected Δt, the process will leave S_2 to enter another state.

Thus, for any Markov process with exponential holding times whose recurrent states form a single chain, we may obtain the limiting-state probabilities by solving these m simultaneous equations.

Again, there exists the important case of birth-and-death processes for which the equations for the limiting-state probabilities are solved with particular ease.

A continuous birth-and-death process is a discrete-state continuous-transition Markov process which obeys the constraint

$$\lambda_{ij} = 0 \qquad \text{if } j \neq i - 1, \quad i + 1$$

(Recall that λ_{ii} has not been defined for continuous-transition processes.) The parameters of the process may be written in a birth-and-death notation and interpreted as follows

$$\lambda_{i(i+1)} = b_i = \textit{average birth rate} \text{ when process is in state } S_i$$

$$\lambda_{i(i-1)} = d_i = \textit{average death rate} \text{ when process is in state } S_i$$

Either by direct substitution into the simultaneous equations or by arguing that, when a birth-and-death process is in the steady state (i.e., the limiting-state probabilities apply), the process must be as likely, in any randomly selected Δt, to undergo an $S_i \to S_{i+1}$ transition as to undergo the corresponding $S_{i+1} \to S_i$ transition, we obtain

$$P_i b_i = P_{i+1} d_{i+1}$$

Thus, for a continuous birth-and-death process, the limiting-state probabilities are found from the simple relations

$$P_{i+1} = \frac{b_i}{d_{i+1}} P_i \qquad \text{and} \qquad \sum_i P_i = 1$$

Several examples with solutions are presented in Sec. 5-8. A wider range of applications is indicated by the set of problems at the end of this chapter. Example 2 in Sec. 5-8 introduces some elementary topics from the theory of queues.

5-8 Examples Involving Continuous-transition Processes

example 1 A young brown-and-white rabbit, named Peter, is hopping about his newly leased two-room apartment. From all available information, we conclude that with probability $\mathcal{P}$ he was in room 1 at $t = 0$. Whenever Peter is in room 1, the probability that he will enter room 2 in the next Δt is known to be equal to $\lambda_{12}\,\Delta t$. At any time when he is in room 2, the probability that he will enter room 1 in the next Δt is $\lambda_{21}\,\Delta t$. It is a bright, sunny day, the wind is 12 mph from the northwest (indoors!), and the inside temperature is 70°F.

(a) Determine $P_1(t)$, the probability that Peter is in room 1 as a function of time for $t \geq 0$.

(b) If we arrive at a random time with the process in the steady state:

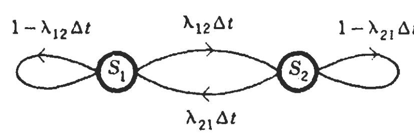
(i) What is the probability the first transition we see will be Peter entering room 2 from room 1?

(ii) What is the probability of a transition occurring in the first Δt after we arrive?

(iii) Determine the PDF $f_x(x_0)$, where x is defined to be the waiting time from our arrival until Peter's next change of room.

(iv) If we observe no transition during the first T units of time after we arrive, what is then the conditional probability that Peter is in room 1?

Let state S_n represent the event "Peter is in room n." We have a two-state Markov process with exponential holding times. We sketch a transition diagram of the process, labeling the branches with the conditional-transition probabilities for trials in an incremental interval Δt.



a It happens that, in this example, the recurrent states form a single chain. However, since we shall solve the general differential equations, we are not taking any steps which require this condition.

$m-1$ eqs.: $P_1(t+\Delta t) = P_1(t)(1-\lambda_{12}\,\Delta t) + P_2(t)\lambda_{21}\,\Delta t$

mth eq.: $\quad 1 = P_1(t) + P_2(t)$

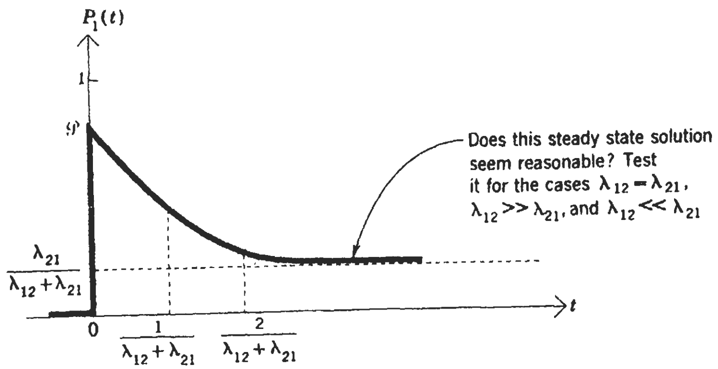
and we have the initial conditions $\underbrace{P(0)} = \underbrace{P_1(0)\ \ P_2(0)} = \underbrace{\mathcal{P}\ \ 1-\mathcal{P}}$.
After collecting terms, dividing both sides of the first equation by Δt, taking the limit as $\Delta t \to 0$, and substituting the second equation into the first equation, we have

$$\frac{dP_1(t)}{dt} + (\lambda_{21}+\lambda_{12})P_1(t) = \lambda_{21} \qquad P_1(0) = \mathcal{P}$$

which is a first-order linear differential equation which has the complete solution

$$P_1(t) = \left(\mathcal{P} - \frac{\lambda_{21}}{\lambda_{21}+\lambda_{12}}\right)e^{-(\lambda_{12}+\lambda_{21})t} + \frac{\lambda_{21}}{\lambda_{12}+\lambda_{21}} \qquad t \geq 0$$

(For readers unfamiliar with how such an equation is solved, this knowledge is not requisite for any other work in this book.) We sketch $P_1(t)$ for a case where $\mathcal{P}$ is greater than $\dfrac{\lambda_{21}}{\lambda_{12}+\lambda_{21}}$.



Since we do happen to be dealing with a birth-and-death process which satisfies the ergodicity condition, we may also obtain the limiting-state probabilities from

$$P_2\lambda_{21} = P_1\lambda_{12} \qquad \text{and} \qquad P_1 + P_2 = 1$$

which does yield the same values for the limiting-state probabilities as those obtained above.

bi The first transition we see will be an $S_1 \to S_2$ transition if and only if Peter happened to be in room 1 when we arrive. Thus, our answer

is simply

$$P_1 = \lambda_{21}(\lambda_{12}+\lambda_{21})^{-1}$$

Note that, although we are equally likely to observe an $S_1 \to S_2$ or an $S_2 \to S_1$ transition in the *first* Δt after we arrive at a random time, it need not follow that the *first transition* we observe is equally likely to be either type. Outcomes of trials in successive Δt's after we arrive at a random time are not independent. For instance, if $\lambda_{12} > \lambda_{21}$ and we arrive at a random instant and wait a very long time without noting any transitions, the conditional probability that the process is in state S_2 approaches unity. We'll demonstrate this phenomenon in the last part of this problem.

bii The probability of a transition in the first Δt after we arrive is simply

$$P_1\lambda_{12}\,\Delta t + P_2\lambda_{21}\,\Delta t = 2\left(\frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+\lambda_{21}}\right)\Delta t$$

which is the sum of the probabilities of the two mutually exclusive ways Peter may make a transition in this first Δt. The quantity $2\lambda_{12}\lambda_{21}(\lambda_{12}+\lambda_{21})^{-1}$ may be interpreted as the average rate at which Peter changes rooms. Of course, the two terms added above are equal, since the average rate at which he is making room $1 \to$ room 2 transitions must equal the average rate at which he makes the only other possible type of transitions. If $\lambda_{12} > \lambda_{21}$, it is true that Peter makes transitions more frequently when he is in room 1 than he does when he is in room 2, but the average transition rates over all time come out equal because he would be in room 1 much less often than he would be in room 2.

biii $f_x(x_0) = P_1 f_{x|S_1}(x_0 \mid S_1) + P_2 f_{x|S_2}(x_0 \mid S_2)$.

$$= \frac{\lambda_{21}}{\lambda_{12}+\lambda_{21}}\lambda_{12}e^{-\lambda_{12}x_0} + \frac{\lambda_{12}}{\lambda_{12}+\lambda_{21}}\lambda_{21}e^{-\lambda_{21}x_0} \qquad x_0 \geq 0$$

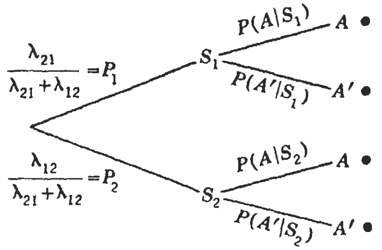
$$= \frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+\lambda_{21}}(e^{-\lambda_{12}x_0} + e^{-\lambda_{21}x_0}) \qquad x_0 \geq 0$$

This answer checks out if $\lambda_{12} = \lambda_{21}$, and furthermore we note that, if $\lambda_{21} \gg \lambda_{12}$, we are almost certain to find the process in state 1 in the steady state and to have the PDF until the next transition be

$$\lambda_{12}e^{-\lambda_{12}x_0} \qquad x_0 \geq 0$$

and the above answer does exhibit this behavior.

biv Define event A: "No transition in first T units of time after we arrive." We may now sketch a sequential event space for the experiment in which we arrive at a random time, given the process is in the steady state.



The quantity $P(A \mid S_1)$ is the conditional probability that we shall see no transitions in the first T units of time after we arrive, given that the process is in state S_1. This is simply the probability that the holding time in S_1 after we arrive is greater than T.

$$P(A \mid S_1) = \int_{t_0=T}^{\infty} \lambda_{12}e^{-\lambda_{12}t_0}\,dt_0 = e^{-\lambda_{12}T} \qquad P(A \mid S_2) = e^{-\lambda_{21}T}$$

We wish to obtain the conditional probability that we found the process in state S_1, given there were no changes of state in the first T units of time after we arrived at a random instant.

$$P(S_1 \mid A) = \frac{P(S_1A)}{P(A)} = \frac{\lambda_{21}e^{-\lambda_{12}T}}{\lambda_{21}e^{-\lambda_{12}T} + \lambda_{12}e^{-\lambda_{21}T}} \qquad T \geq 0$$

This answer checks out for $\lambda_{12} = \lambda_{21}$ and as $T \to 0$. Furthermore, if $\lambda_{21} \gg \lambda_{12}$, we would expect that, as $T \to \infty$, we would be increasingly likely to find the process in its slow transition (long-holding-time) state and our answer does exhibit this property.

example 2 Consider a service facility at which the arrival of customers is a Poisson process, with an average arrival rate of λ customers per hour. If customers find the facility fully occupied when they arrive, they enter a *queue* (a waiting line) and await their turns to be serviced on a first-come first-served basis. Once a customer leaves the queue and enters actual service, the time required by the facility to service him is an independent experimental value of an exponentially distributed random variable with an expected value of μ^{-1} hours ($\mu > \lambda$).

Determine the limiting-state probabilities and the expected value for the *total* number of customers at the facility (in the queue and in service) if

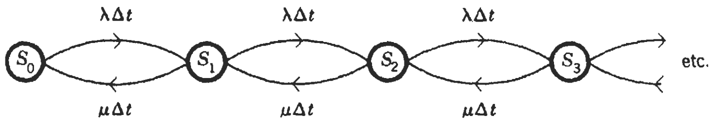
(a) The facility can service only one customer at a time.
(b) The facility can service up to an infinite number of customers in parallel.

We shall study several other aspects of these situations as we answer the above questions. To begin our solution, we define the event S_i by

S_i: There are a total of i customers at the facility, where i includes both customers in the queue and those receiving service.

Since the customer arrival rate for either case is independent of the state of the system, the probability of an $S_i \to S_{i+1}$ transition in any incremental interval Δt is equal to $\lambda\,\Delta t$. The probability of completing a service and having an arrival in the same Δt is a second-order term.

a We are told that the service times are independent exponential random variables with the PDF $f_x(x_0) = \mu e^{-\mu x_0}$ for $x_0 \geq 0$. Parameter μ represents the maximum possible service rate at this facility. *If there were always at least one customer at the facility*, the completion of services would be a Poisson process with an average service completion rate of μ services per hour. Considering only first-order terms and not bothering with the *self-loops*, we have the state transition diagram



We are concerned with an infinite-state single-chain continuous-birth-and-death process, with

$$b_i = \lambda \qquad i = 0, 1, 2, \ldots \qquad d_i = \begin{cases} 0 & i = 0 \\ \mu & i = 1, 2, 3, \ldots \end{cases}$$

For the case of interest, $\mu > \lambda$, we shall assume that the limiting-state probabilities exist for this infinite-state process. If the maximum service rate were *less* than the average arrival rate, we would expect the length of the line to become infinite. We use the relations

$$P_{i+1} = \frac{b_i}{d_{i+1}} P_i \qquad i = 0, 1, 2, \ldots \qquad \text{and} \qquad \sum_i P_i = 1$$

and there follow

$$P_1 = \frac{\lambda}{\mu} P_0 \qquad P_2 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0 \qquad P_3 = \frac{\lambda}{\mu} P_2 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$P_i = \left(\frac{\lambda}{\mu}\right)^i P_0 \qquad \sum_i P_i = 1 = \sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i P_0 = \left(1 - \frac{\lambda}{\mu}\right)^{-1} P_0$$

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$P_i = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i \qquad i = 0, 1, 2, \ldots; \qquad \mu > \lambda$$

$$E(i) = \sum_i iP_i = \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right)^{-1} \qquad \mu > \lambda$$

The expected value of the total number of customers at the facility when the process is in the steady state, $E(i)$, is obtained above by either using z transforms or noting the relation of the P_i's to the geometric PMF.

It is interesting to observe, for instance, that, if the average arrival rate is only 80% of the maximum average service rate, there will be, on the average, a total of four customers at the facility. When there are four customers present, three of them will be *waiting* to enter service even though the facility is empty 20% of all time. Such is the price of randomness.

Let's find the PDF for t, the total time (waiting for service and during service) spent by a randomly selected customer at this single-channel service facility. Customers arrive randomly, and the probability any customer will find exactly i *other* customers already at the facility is P_i. If he finds i other customers there already, a customer will leave after a total of $i + 1$ independent exponentially distributed service times are completed. Thus the conditional PDF for the waiting time of this customer is an Erlang PDF (Sec. 4-6) of order $i + 1$, and we have

$$f_t(t_0) = \sum_i f_{t|S_i}(t_0 \mid S_i)P(S_i) = \sum_i \frac{\mu^{i+1}t_0^i e^{-\mu t_0}}{i!} P_i$$

$$= \sum_{i=0}^{\infty} \frac{\mu^{i+1}t_0^i e^{-\mu t_0}}{i!} \left(\frac{\lambda}{\mu}\right)^i \left(1 - \frac{\lambda}{\mu}\right)$$

$$= (\mu - \lambda)e^{-(\mu-\lambda)t_0} \qquad t_0 \geq 0; \qquad \mu > \lambda$$

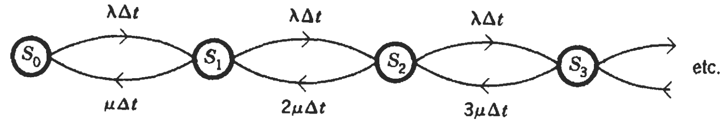
Thus, the total time any customer spends at the facility, for a process with independent exponentially distributed interarrival times and service times, is also an exponential random variable. The expected time spent at the facility is then

$$E(t) = (\mu - \lambda)^{-1} \qquad \mu > \lambda$$

and we can check this result by also using our result from Sec. 3-7, noticing that t is the sum of a random number $i + 1$ of independent service times,

$$E(t) = E(i + 1)E(x) = \left[\frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right)^{-1} + 1\right]\frac{1}{\mu} \stackrel{\checkmark}{=} (\mu - \lambda)^{-1}$$

b For this part, we consider the case where there are an infinite number of parallel service stations available at the facility and no customer has to wait before entering service. When there are exactly i customers in service at this facility, the probability that any particular customer will leave (complete service) in the next Δt is $\mu\,\Delta t$. *While there are i customers present*, we are concerned with departures representing services in any of i independent such processes and, to the first order, the probability of one departure in the next Δt is $i\mu\,\Delta t$. We have, again omitting the self-loops in the transition diagram,



$b_i = \lambda \quad i = 0, 1, 2 \ldots \qquad d_i = i\mu \quad i = 0, 1, 2 \ldots$

Use of our simplified procedures for obtaining the limiting-state probabilities for birth-and-death processes proceeds:

$$P_{i+1} = \frac{b_i}{d_{i+1}} P_i \qquad i = 0, 1, 2, \ldots \qquad \sum_i P_i = 1$$

$$P_1 = \frac{\lambda}{\mu} P_0 \qquad P_2 = \frac{\lambda}{2\mu} P_1 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 P_0 \qquad P_3 = \frac{\lambda}{3\mu} P_2 = \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$P_i = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i P_0 \qquad \sum_i P_i = 1 = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i P_0 = e^{\lambda/\mu}P_0 \qquad P_0 = e^{-\lambda/\mu}$$

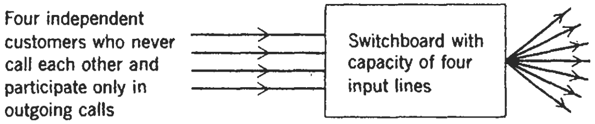
$$P_i = \left(\frac{\lambda}{\mu}\right)^i \frac{e^{-\lambda/\mu}}{i!} \qquad i = 0, 1, 2, 3, \ldots \qquad E(i) = \frac{\lambda}{\mu} \qquad \mu > \lambda > 0$$

The limiting-state probabilities for this case form a Poisson PMF for the total number of customers at the facility (all of whom are in service) at a random time. As one would expect, P_0 is greater for this case than in part (a), and all other P_i's here are less than the corresponding quantities for that single-channel case. For instance, if $\lambda/\mu = 0.8$, this facility is completely idle a fraction

$$P_0 = e^{-(4/5)} = 0.45$$

of all time. This compares with $P_0 = 0.20$ for the same λ/μ ratio for the single-channel case in part (a).

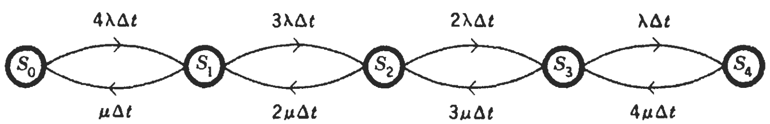
example 3 A four-line switchboard services outgoing calls of four subscribers who never call each other. The durations of all phone calls are independent identically distributed exponential random variables with an expected value of μ^{-1}. For each subscriber, the interval between the end of any call and the time he places his next call is an independent exponential random variable with expected value λ^{-1}.



We shall assume that the system is in the steady state, neglect the possibility of busy signals, and use the state notation S_i: Exactly i of the input lines are active.

(a) Determine the limiting-state probabilities for the number of the input lines in use at any time.
(b) Given that there are at present three switchboard lines in use, determine the PDF for the waiting time until the next change of state.
(c) Determine the expected value of the number of busy input lines.
(d) What is the probability that there are exactly two lines busy at the switchboard the instants just before the arrivals of both members of a randomly selected pair of successive outgoing calls?

a Reasoning similar to that used in part (*b*) of the previous example leads us to the following observation. Given that there are exactly i input lines in use at the present time, the conditional probability of an $S_i \to S_{i+1}$ transition in the next Δt is $(4 - i)\lambda\,\Delta t$, and the conditional probability of an $S_i \to S_{i-1}$ transition in the next Δt is $i\mu\,\Delta t$. Neglecting the self-loops, we have the transition diagram



	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$
b_i	4λ	3λ	2λ	λ	0
d_i	0	μ	2μ	3μ	4μ

This is a single-chain process with no periodicities; so the limiting-state probabilities will not depend on the initial conditions. Furthermore, it is a birth-and-death process; so we may use the special form of the simultaneous equations for the limiting-state probabilities,

$$P_{i+1} = \frac{b_i}{d_{i+1}} P_i \qquad i = 0, 1, 2, 3, 4 \qquad \sum_{i=0}^{4} P_i = 1$$

which result in

$$P_0 = \left(1 + \frac{\lambda}{\mu}\right)^{-4} \qquad P_1 = 4\frac{\lambda}{\mu}\left(1 + \frac{\lambda}{\mu}\right)^{-4} \qquad P_2 = 6\left(\frac{\lambda}{\mu}\right)^2\left(1 + \frac{\lambda}{\mu}\right)^{-4}$$

$$P_3 = 4\left(\frac{\lambda}{\mu}\right)^3\left(1 + \frac{\lambda}{\mu}\right)^{-4} \qquad P_4 = \left(\frac{\lambda}{\mu}\right)^4\left(1 + \frac{\lambda}{\mu}\right)^{-4}$$

As we would expect, if $\lambda \gg \mu$, P_4 is very close to unity and, if $\lambda \ll \mu$, P_0 is very close to unity.

b Given that the system is in state S_3, the probability that it will leave this state in the next Δt is $(3\mu + \lambda)\,\Delta t$, no matter how long it has been in S_3. Thus, the PDF for t, the exponential holding time in this state, is

$$f_t(t_0) = (3\mu + \lambda)e^{-(3\mu+\lambda)t_0} \qquad t_0 \geq 0$$

c Direct substitution of the P_i's obtained in part (*a*) into

$$E(i) = \sum_{i=0}^{4} iP_i$$

results in

$$E(i) = 4\frac{\lambda}{\mu}\left(1 + \frac{\lambda}{\mu}\right)^{-1}$$

and this answer agrees with our intuition for $\lambda \gg \mu$ and $\mu \gg \lambda$.

d This is as involved a problem as one is likely to encounter. Only the answer and a rough outline of how it may be obtained are given. The serious reader should be sure that he can supply the missing steps; the less serious reader may choose to skip this part.

$$\text{Answer} = \left(\frac{2\lambda P_2}{4\lambda P_0 + 3\lambda P_1 + 2\lambda P_2 + \lambda P_3}\right) \times \left(\frac{3\mu}{\lambda + 3\mu}\right) \times \left(\frac{\lambda}{\lambda + \mu}\right)$$

Probability first member of our pair of arrivals becomes 3d customer present at switchboard

Conditional probability that next change of state is due to completion of a call, given system is in state S_3

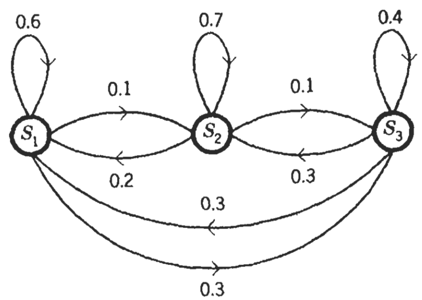
Conditional probability that next change of state is due to an arrival, given system is in state S_2

We have assumed that the "randomly selected" pair was chosen by selecting the first member of the pair by means of an equally likely choice among a large number of incoming calls. If the first member were to be chosen by selecting the first incoming call to follow a randomly selected instant, we would have a different situation.

PROBLEMS

5.01 For a series of dependent trials, the probability of success on any trial is given by $(k + 1)/(k + 3)$, where k is the number of successes in the previous three trials. Define a state description and set of transition probabilities which allow this process to be described as a Markov process. Draw the state transition diagram. Try to use the smallest possible number of states.

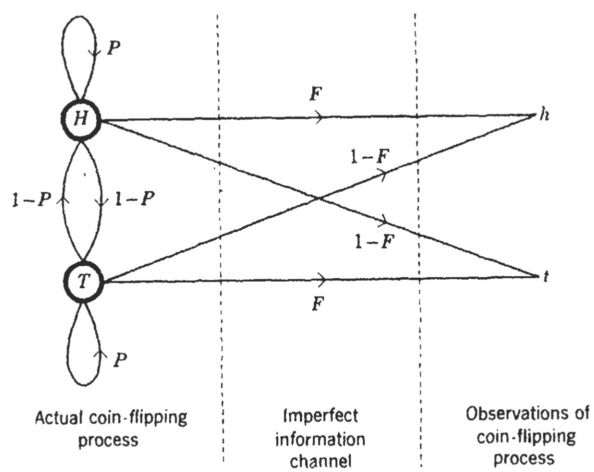
5.02 Consider the three-state discrete-transition Markov process



Determine the three-step transition probabilities $p_{11}(3)$, $p_{12}(3)$, and $p_{13}(3)$ both from a sequential sample space and by using the equation $p_{ij}(n+1) = \sum_k p_{ik}(n)p_{kj}$ in an effective manner.

5.03 We are observing and recording the outcomes of dependent flips of a coin at a distance on a foggy day. The probability that any flip will have the same outcome as the previous flip is equal to P. Our observations of the experimental outcomes are imperfect. In fact, the probability that we shall properly record the outcome of any trial is equal to F and is independent of all previous or future errors. We use the notation

h_n: We record the observation of the nth trial to be heads.
t_n: We record the observation of the nth trial to be tails.



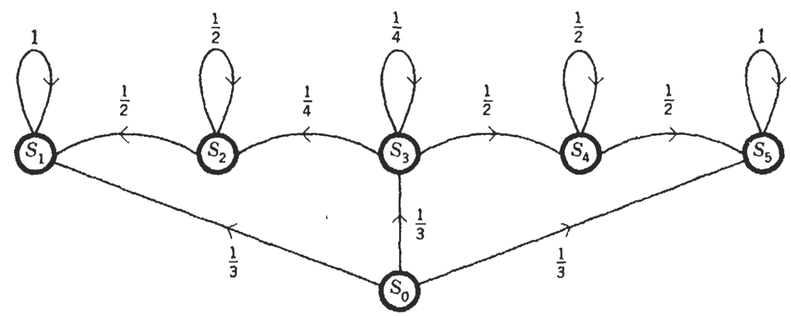
Can the possible sequences of our observations be modeled as the state history of a two-state Markov process?

5.04 **a** Identify the transient, recurrent, and periodic states of the discrete-state discrete-transition Markov process described by

$$[p] = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0.6 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.3 & 0.4 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

b How many chains are formed by the recurrent states of this process?
c Evaluate $\lim_{n\to\infty} p_{41}(n)$ and $\lim_{n\to\infty} p_{66}(n)$.

5.05 For the Markov process pictured here, the following questions may be answered by inspection:



Given that this process is in state S_0 just before the first trial, determine the probability that:
a The process enters S_2 for the first time as the result of the Kth trial.
b The process never enters S_4.
c The process does enter S_2, but it also leaves S_2 on the trial after it entered S_2.
d The process enters S_1 for the first time on the third trial.
e The process is in state S_3 immediately after the Nth trial.

5.06 Days are either good (G), fair (F), or sad (S). Let F_n, for instance, be the event that the nth day is fair. Assume that the probability of having a good, fair, or sad day depends only on the condition of the previous day as dictated by the conditional probabilities

$P(F_{n+1} \mid G_n) = 1/3$ $P(F_{n+1} \mid S_n) = 3/8$ $P(S_{n+1} \mid F_n) = 1/6$
$P(F_{n+1} \mid F_n) = 2/3$ $P(S_{n+1} \mid G_n) = 1/6$ $P(S_{n+1} \mid S_n) = 1/2$

Assume that the process is in the steady state. A good day is worth \$1, a fair day is worth \$0, and a sad day is worth −\$1.

a Determine the expected value and the variance of the value of a randomly selected day.
b Determine the expected value and the variance of the value of a random two-day sequence. Compare with the above results, and comment.
c With the process in the steady-state at day zero, we are told that the sum value of days 13 and 14 was $0. What is the probability that day 13 was a fair day?

5.07 The outcomes of successive flips of a particular coin are dependent and are found to be described fully by the conditional probabilities

$$\text{Prob}(H_{n+1} \mid H_n) = 3/4 \qquad \text{Prob}(T_{n+1} \mid T_n) = 2/3$$

where we have used the notation
Event H_k: Heads on kth toss   Event T_k: Tails on kth toss

We know that the first toss came up heads.
a Determine the probability that the *first* tail will occur on the kth toss ($k = 2, 3, 4, \ldots$).
b What is the probability that flip 5,000 will come up heads?
c What is the probability that flip 5,000 will come up heads and flip 5,002 will also come up heads?
d Given that flips 5,001, 5,002, . . . , 5,000 + m all have the same result, what is the probability that all of these m outcomes are heads? Simplify your answer as much as possible, and interpret your result for large values of m.
e We are told that the 375th head just occurred on the 500th toss. Determine the expected value of the number of additional flips required until we observe the 379th head.

5.08 In his new job for the city, Joe makes daily measurements of the level of the Sludge River. His predecessor established that these daily readings can be modeled by a Markov process and that there are only three possible river depths, zero, one, and two feet. It is also known that the river level never changes more than one foot per day. The city has compiled the following transition probabilities:

$$p_{01} = 1/4 \qquad p_{10} = 1/2 \qquad p_{12} = 1/4 \qquad p_{22} = 1/4$$

Let x_K represent Joe's reading of the river depth on his Kth day on the job. We are given that the reading on the day before he started was one foot.

Determine:
a The probability mass function for random variable x_1
b The probability that $x_{377} \neq x_{378}$
c The conditional probability mass function for x_{999}, given that $x_{1,000} = 1$

d The numerical values of
i $\lim_{n\to\infty} E(x_{n+1} - x_n)$ **ii** $\lim_{n\to\infty} E[(x_{n+1} - x_n)^2]$
iii $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i$
e The z transform of the probability mass function $p_L(L_0)$, where random variable L is the smallest positive integer which satisfies $x_L \neq x_1$.

5.09 Mr. Mean Variance has the only key which locks or unlocks the door to Building 59, the Probability Building. He visits the door once each hour on the hour. When he arrives:
If the door is open, he locks it with probability 0.3.
If the door is locked, he unlocks it with probability 0.8.
a After he has been on the job several months, is he more likely to lock the door or to unlock it on a randomly selected visit?
b With the process in the steady state, Joe arrived at Building 59 two hours ahead of Harry. What is the probability that each of them found the door in the same condition?
c Given the door was open at the time Mr. Variance was hired, determine the z transform for the number of visits up to and including the one on which he unlocked the door himself for the first time.

5.10 Arrivals of potential customers on the street outside a one-pump gas station are noted to be a Poisson process with an average arrival rate of λ cars per hour. Potential customers will come in for gas if there are fewer than two cars already at the pump (including the one being attended to). If there are two cars already at the pump, the potential customers will go elsewhere.

It is noted that the amount of time required to service any car is an independent random variable with PDF

$$f_T(T_0) = \begin{cases} \mu e^{-\mu T_0} & T_0 \geq 0 \\ 0 & T_0 < 0 \end{cases}$$

a Give a physical interpretation of the constant μ.
b Write the differential equations relating the $P_n(t)$'s, where $P_n(t)$ is the probability of having n cars at the pump at time t. Do not solve the equations.
c Write and solve the equations for P_n, $n = 0, 1, 2, 3, \ldots$, where P_n is the steady-state probability of having a total of n cars at the pump.
d If the cars arrive at the average rate of 20 per hour and the average service time is two minutes per car, what is the probability that a potential customer will go elsewhere? What fraction of the attendant's time will be spent servicing cars?

e At the same salary, the owner can provide a more popular, but slower, attendant. This would raise the average service time to 2.5 minutes per car but would also increase λ from 20 to 28 cars per hour. Which attendant should he use to maximize his expected profit? Determine the percent change in the number of customers serviced per hour that would result from changing to the slower attendant.

5.11 At a single service facility, the interarrival times between successive customers are independent exponentially distributed random variables. The average customer arrival rate is 40 customers per hour.

When a total of two or fewer customers are present, a single attendant operates the facility and the service time for each customer is an exponentially distributed random variable with a mean value of two minutes.

Whenever there are three or more customers at the facility, the attendant is joined by an assistant. In this case, the service time is an exponentially distributed random variable with an expected value of one minute.

Assume the process is in the steady state.

a What fraction of the time are both of them free?

b What is the probability that both men are working at the facility the instant before a randomly selected customer arrives? The instant after he arrives?

c Each of the men is to receive a salary proportional to the expected value of the amount of time he is actually at work servicing customers. The constant of proportionality is the same for both men, and the sum of their salaries is to be \$100. Determine the salary of each man.

5.12 Only two taxicabs operate from a particular station. The total time it takes a taxi to service any customer and return to the station is an exponentially distributed random variable with a mean of $1/\mu$ hours. Arrivals of potential customers are modeled as a Poisson process with average rate of λ customers per hour. If any potential customer finds no taxi at the station at the instant he arrives, he walks to his destination and thus does not become an actual customer. The cabs always return directly to the station without picking up new customers. All parts of this problem are independent of statements in other parts.

a If $\lambda = \infty$, determine (in as simple and logical a manner as you can) the average number of customers served per hour.

b Using the notation $\rho = \mu/\lambda$, determine the steady-state probability that there is exactly one taxi at the station.

c If we survey a huge number of actual customers, what fraction of them will report that they arrived at an instant when there was exactly one taxi at the station?

d Taxi A has been destroyed in a collision with a Go-Kart, and we note that B is at the station at time $t = 0$. What are the expected value and variance of the time until B leaves the station with his fourth fare since $t = 0$?

5.13 All ships travel at the same velocity through a wide canal. Eastbound ship arrivals at the canal are a Poisson process with an average arrival rate λ_E ships per day. Westbound ships arrive as an independent Poisson process with average arrival rate λ_W per day. An indicator at a point in the canal is always pointing in the direction of travel of the most recent ship to pass it. Each ship takes T days to traverse the canal. Use the notation $\rho = \lambda_E/\lambda_W$ wherever possible.

a What is the probability that the next ship passing by the indicator causes it to change its indicated direction?

b What is the probability that an eastbound ship will see no westbound ships during its eastward journey through the canal?

c If we begin observing at an arbitrary time, determine the probability mass function $p_k(k_0)$, where k is the total number of ships we observe up to and including the seventh eastbound ship we see.

d If we begin observing at an arbitrary time, determine the probability density function $f_t(t_0)$, where t is the time until we see our seventh eastbound ship.

e Given that the pointer is pointing west:

i What is the probability that the next ship to pass it will be westbound?

ii What is the probability density function for the remaining time until the pointer changes direction?

5.14 A switchboard has two outgoing lines and is concerned only with servicing the outgoing calls of three customers who never call each other. When he is not on a line, each potential caller generates calls at a Poisson rate λ. Call lengths are exponentially distributed, with a mean call length of $1/\mu$. If a caller finds the switchboard blocked, he never tries to reinstitute that particular call.

a Determine the fraction of time that the switchboard is saturated.

b Determine the fraction of outgoing calls which encounter a saturated switchboard.

5.15 An illumination system contains $R + 3$ bulbs, each of which fails independently and has a life span described by the probability density function

$$f_t(t_0) = \lambda e^{-\lambda t_0} \qquad t_0 \geq 0$$

At the time of the third failure, the system is shut down, the dead bulbs are replaced, and the system is "restarted." The down time for the service operation is a random variable described by the probability density function

$$f_x(x_0) = \mu^2 x_0 e^{-\mu x_0} \qquad x_0 \geq 0$$

a Determine the mean and variance of y, the time from $t = 0$ until the end of the kth service operation.
b Determine the steady-state probability that all the lights are on.

5.16 Potential customers arrive at the input gate to a facility in a Poisson manner with average arrival rate λ. The facility will hold up to three customers including the one being serviced. Potential customers who arrive when the facility is full go elsewhere for service. Service time is an exponential random variable with mean $1/\mu$. Customers leave as soon as they are serviced. Service for actual customers is on a first-come first-served basis.

a If we select a random pair (see final comment in Sec. 5-8) of successive *potential* customers approaching the facility, what is the probability that they will eventually emerge as a pair of successive *actual* customers?
b If we select a random pair of successive *actual* customers leaving the facility, what is the probability that these customers arrived as successive *potential* customers at the facility input?
c Starting at a randomly selected time, what is the probability that *before the next actual customer arrives at the input gate* at least two customers would be observed leaving via the output gate?

5.17 Consider a K-state discrete-state Markov system with exponential holding times. The system is composed of a single chain.

$\lambda_{ij}\,\Delta t$ ($i \neq j$) = conditional probability that system will enter state S_j in the next Δt, given that present state is S_i

Other than in part (a) you may use the limiting-state probabilities as $P_1, P_2, \ldots, P_K$ in your answers.

a Write, in a simple form, a set of equations which determine the steady-state probabilities $P_1, P_2, \ldots, P_K$.
b Given that the process is at present in state S_i, what is the probability that two transitions from now it will again be in state S_i?
c What are the expected value and variance of the time the system spends during any visit to state S_i?
d Determine the average rate at which the system makes transitions.
e If we arrive at a random time with the process in the steady state, determine the probability density function for the time until the next transition after we arrive.

f Suppose that the process is a pure birth-and-death process and we arrive at a random time. What is the probability that the first transition we observe will be due to a birth? That both of the first two transitions we observe will be due to births?

5.18 An information source is always in one of m mutually exclusive, collectively exhaustive states $S_1, S_2, \ldots, S_m$. Whenever this source is in state S_i:

1 It produces printed messages in a Poisson manner at an average rate of μ_i messages per hour.
2 The conditional probability that the source will enter state S_j ($j \neq i$) in the next incremental Δt is given by $\lambda_{ij}\,\Delta t$.

All messages are numbered consecutively and filed in a warehouse. The process is in the steady state and you may assume that the limiting state probabilities for the source, $P_1, P_2, \ldots, P_m$, are known quantities.

Each part of this problem is to be worked separately.

a Given that the process has been in state S_2 for the last three hours what is the probability that no messages were produced in the last 1.5 hours?
b Given that the process is *not* in state S_2, what is the probability that it will enter S_2 in the next incremental Δt?
c Determine the average rate at which messages are produced.
d What is the probability that the source will produce exactly two messages during any particular visit to state S_2?
e If we arrive at a random time to observe the process, what is the probability that we see at least one message generated before we observe the next state transition of the message source?
f If we select a random message from the file in the warehouse, what is the probability that it was produced when the source was in state S_2?
g If we select a pair of *consecutive* messages at random from the file, what is the probability that the source underwent exactly one change of state during the interval between the instants at which these two messages were produced?
h If we are told that, during the last 10 hours, the source process underwent exactly eight changes of state and spent
Exactly two hours in state S_3
Exactly five hours in state S_7
Exactly three hours in state S_8
determine the exact conditional PMF for the number of messages produced during the last 10 hours.