Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Recitation 8 Solutions

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1. (a) We know that the PDF must integrate to 1. Therefore we have

$$\int_{-\infty}^{\infty} f_Z(z)\,dz \;=\; \int_{-2}^{1} \gamma(1+z^2) \;=\; \gamma\left(z+\frac{1}{3}z^3\right)\Big|_{-2}^{1} \;=\; 6\gamma.$$

From this we conclude $\gamma = 1/6$.

(b) To find the CDF, we integrate:

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{z} f_Z(t)\,dt = \begin{cases} 0, & \text{if } z < -2, \\ \frac{1}{6}\left(t+\frac{1}{3}t^3\right)\big|_{-2}^{z}, & \text{if } -2 \le z \le 1, \\ 1, & \text{if } z > 1 \end{cases} \\ &= \begin{cases} 0, & \text{if } z < -2, \\ \frac{1}{6}(z+\frac{1}{3}z^3+\frac{14}{3}), & \text{if } -2 \le z \le 1, \\ 1, & \text{if } z > 1. \end{cases} \end{aligned}$$

2. See textbook, Problem 3.9, page 187.

3. (a) For $x \ge 0$,

$$F_X(x) \;=\; \int_{-\infty}^{x} f_X(t)\,dt \;=\; \int_{0}^{x} \lambda e^{-\lambda t}\,dt \;=\; \left[-e^{-\lambda t}\right]_0^x \;=\; 1-e^{-\lambda x}.$$

For $x < 0$, we have $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = 0$. Thus we conclude

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1-e^{-\lambda x}, & \text{if } x \ge 0. \end{cases}$$

(b) The key step in the following computation uses integration by parts, whereby

$$\int_0^\infty u\,dv \;=\; uv\Big|_0^\infty - \int_0^\infty v\,du$$

is applied with $u = x$ and $v = -e^{-\lambda x}$:

$$\mathbf{E}[X] \;=\; \int_{-\infty}^{\infty} x f_X(x)\,dx \;=\; \int_0^\infty x\lambda e^{-\lambda x}\,dx \;=\; \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\,dx \;=\; \frac{1}{\lambda}.$$

(c) Integrating by parts with $u = x^2$ and $v = -e^{-\lambda x}$ in the second line below gives

$$\begin{aligned} \mathbf{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \;=\; \int_0^\infty x^2\lambda e^{-\lambda x}\,dx \\ &= \left[-x^2e^{-\lambda x}\right]_0^\infty + 2\int_0^\infty xe^{-\lambda x}\,dx \;=\; \frac{2}{\lambda}\mathbf{E}[X] \;=\; \frac{2}{\lambda^2}. \end{aligned}$$

Combining with the previous computation, we obtain

$$\text{var}(X) \;=\; \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \;=\; \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \;=\; \frac{1}{\lambda^2}.$$

(d) The maximum of a set is upper bounded by z when each element of the set is upper bounded by z. Thus for any positive z,

$$\begin{aligned}
\mathbf{P}(Z \le z) &= \mathbf{P}(\max\{X_1, X_2, X_3\} \le z) = \mathbf{P}(X_1 \le z,\, X_2 \le z,\, X_3 \le z) \\
&= \mathbf{P}(X_1 \le z)\,\mathbf{P}(X_2 \le z)\,\mathbf{P}(X_3 \le z) \\
&= (1 - e^{-\lambda z})^3,
\end{aligned}$$

where the third equality uses the independence of X_1, X_2, and X_3. Thus,

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ (1 - e^{-\lambda z})^3, & \text{if } z \ge 0. \end{cases}$$

Differentiating the CDF gives the desired PDF:

$$f_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ 3\lambda e^{-\lambda z}(1 - e^{-\lambda z})^2, & \text{if } z \ge 0. \end{cases}$$

(e) The minimum of a set is lower bounded by w when each element of the set is lower bounded by w. Thus for any positive w,

$$\begin{aligned}
\mathbf{P}(W \ge w) &= \mathbf{P}(\min\{X_1, X_2\} \ge w) = \mathbf{P}(X_1 \ge w,\, X_2 \ge w) \\
&= \mathbf{P}(X_1 \le w)\,\mathbf{P}(X_2 \le w) \\
&= (e^{-\lambda w})^2 = e^{-2\lambda w}
\end{aligned}$$

where the third equality uses the independence of X_1 and X_2. Thus,

$$F_W(w) = \begin{cases} 0, & \text{if } w < 0, \\ 1 - e^{-2\lambda w}, & \text{if } w \ge 0. \end{cases}$$

We can recognize this as the CDF of an exponential random variable with parameter 2λ. The PDF is

$$f_W(w) = \begin{cases} 0, & \text{if } w < 0, \\ 2\lambda e^{-2\lambda w}, & \text{if } w \ge 0. \end{cases}$$