MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Quiz 2 Solutions | Spring 2008)

Question 1

Multiple choice questions. **CLEARLY** circle the best answer for each question below. Each question is worth 4 points each, with no partial credit given.

a. Let X_1, X_2, and X_3 be independent random variables with the continuous uniform distribution over $[0,1]$. Then $\mathbf{P}(X_1 < X_2 < X_3) =$

(i) $\boxed{1/6}$
(ii) $1/3$
(iii) $1/2$
(iv) $1/4$

Solution: To understand the principle, first consider a simpler problem with X_1 and X_2 as given above. Note that $\mathbf{P}(X_1 < X_2) + \mathbf{P}(X_2 < X_1) + \mathbf{P}(X_1 = X_2) = 1$ since the corresponding events are disjoint and exaust all the possibilities. But $\mathbf{P}(X_1 < X_2) = \mathbf{P}(X_2 < X_1)$ by symmetry. Furthermore, $\mathbf{P}(X_1 = X_2) = 0$ since the random variables are continuous. Therefore, $\mathbf{P}(X_1 < X_2) = 1/2$.

Analogously, omitting the events with zero probability but making sure to exhaust all other possibilities, we have that $\mathbf{P}(X_1 < X_2 < X_3) + \mathbf{P}(X_1 < X_3 < X_2) + \mathbf{P}(X_2 < X_1 < X_3) + \mathbf{P}(X_2 < X_3 < X_1) + \mathbf{P}(X_3 < X_1 < X_2) + \mathbf{P}(X_3 < X_2 < X_1) = 1$. And, by symmetry, $\mathbf{P}(X_1 < X_2 < X_3) = \mathbf{P}(X_1 < X_3 < X_2) = \mathbf{P}(X_2 < X_1 < X_3) = \mathbf{P}(X_2 < X_3 < X_1) = \mathbf{P}(X_3 < X_1 < X_2) = \mathbf{P}(X_3 < X_2 < X_1)$. Thus, $\mathbf{P}(X_1 < X_2 < X_3) = 1/6$.

b. Let X and Y be two continuous random variables. Then

(i) $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$
(ii) $\boxed{\mathbf{E}[X^2 + Y^2] = \mathbf{E}[X^2] + \mathbf{E}[Y^2]}$
(iii) $f_{X+Y}(x+y) = f_X(x)f_Y(y)$
(iv) $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$

Solution: Since X^2 and Y^2 are random variables, the result follows by the linearity of expectation.

c. Suppose X is uniformly distributed over $[0,4]$ and Y is uniformly distributed over $[0,1]$. Assume X and Y are independent. Let $Z = X + Y$. Then

(i) $f_Z(4.5) = 0$
(ii) $\boxed{f_Z(4.5) = 1/8}$
(iii) $f_Z(4.5) = 1/4$
(iv) $f_Z(4.5) = 1/2$

Solution: Since X and Y are independent, the result follows by convolution:

$$f_Z(4.5) = \int_{-\infty}^{\infty} f_X(\alpha) f_Y(4.5 - \alpha)\, d\alpha = \int_{3.5}^{4} \frac{1}{4}\, d\alpha = \frac{1}{8}\,.$$

d. For the random variables defined in part (c), $\mathbf{P}(\max(X,Y) > 3)$ is equal to

(i) 0

(ii) 9/4

(iii) 3/4

(iv) $\boxed{1/4}$

Solution: Note that $\mathbf{P}(\max(X,Y) > 3) = 1 - \mathbf{P}(\max(X,Y) \leq 3) = 1 - \mathbf{P}(\{X \leq 3\} \cap \{Y \leq 3\})$. But, X and Y are independent, so $\mathbf{P}(\{X \leq 3\} \cap \{Y \leq 3\}) = \mathbf{P}(X \leq 3)\mathbf{P}(Y \leq 3)$. Finally, computing the probabilities, we have $\mathbf{P}(X \leq 3) = 3/4$ and $\mathbf{P}(X \leq 3) = 1$. Thus, $\mathbf{P}(\max(X,Y) > 3) = 1 - 3/4 = 1/4$.

e. Recall the hat problem from lecture: N people put their hats in a closet at the start of a party, where each hat is uniquely identified. At the end of the party each person randomly selects a hat from the closet. Suppose N is a Poisson random variable with parameter λ. If X is the number of people who pick their own hats, then $\mathbf{E}[\mathrm{X}]$ is equal to

(i) λ

(ii) $1/\lambda^2$

(iii) $1/\lambda$

(iv) $\boxed{1}$

Solution: Let $X = X_1 + \ldots + X_N$ where each X_i is the indicator function such that $X_i = 1$ if the i^{th} person picks their own hat and $X_i = 0$ otherwise. By the linearity of the expectation, $\mathbf{E}[X \mid N = n] = \mathbf{E}[X_1 \mid N = n] + \ldots + \mathbf{E}[X_N \mid N = n]$. But $\mathbf{E}[X_i \mid N = n] = 1/n$ for all $i = 1, \ldots, n$. Thus, $\mathbf{E}[X \mid N = n] = n\mathbf{E}[X_i \mid N = n] = 1$. Finally, $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid N = n]] = 1$.

f. Suppose X and Y are Poisson random variables with parameters λ_1 and λ_2 respectively, where X and Y are independent. Define $W = X + Y$, then

(i) W is Poisson with parameter $\min(\lambda_1, \lambda_2)$

(ii) $\boxed{W \text{ is Poisson with parameter } \lambda_1 + \lambda_2}$

(iii) W may not be Poisson but has mean equal to $\min(\lambda_1, \lambda_2)$

(iv) W may not be Poisson but has mean equal to $\lambda_1 + \lambda_2$

Solution: The quickest way to obtain the answer is through transforms: $M_X(s) = e^{\lambda_1(e^s-1)}$ and $M_Y(s) = e^{\lambda_2(e^s-1)}$. Since X and Y are independent, we have that $M_W(s) = e^{\lambda_1(e^s-1)}e^{\lambda_2(e^s-1)} = e^{(\lambda_1+\lambda_2)(e^s-1)}$, which equals the transform of a Poisson random variable with mean $\lambda_1 + \lambda_2$.

g. Let X be a random variable whose transform is given by $M_X(s) = (0.4 + 0.6e^s)^{50}$. Then

(i) $\mathbf{P}(X = 0) = \mathbf{P}(X = 50)$

(ii) $\mathbf{P}(X = 51) > 0$

(iii) $\boxed{\mathbf{P}(X = 0) = (0.4)^{50}}$

(iv) $\mathbf{P}(X = 50) = 0.6$

Solution: Note that $M_X(s)$ is the transform of a binomial random variable X with $n = 50$ trials and the probability of success $p = 0.6$. Thus, $\mathbf{P}(X = 0) = 0.4^{50}$.

h. Let X_i, $i = 1, 2, \ldots$ be independent random variables all distributed according to the PDF $f_X(x) = x/8$ for $0 \leq x \leq 4$. Let $S = \frac{1}{100}\sum_{i=1}^{100} X_i$. Then $\mathbf{P}(S > 3)$ is approximately equal to

(i) $1 - \Phi(5)$

(ii) $\Phi(5)$

(iii) $\boxed{1 - \Phi\left(\frac{5}{\sqrt{2}}\right)}$

(iv) $\Phi\left(\frac{5}{\sqrt{2}}\right)$

Solution: Let $S = \frac{1}{100}\sum_{i=1}^{100} Y_i$ where Y_i is the random variable given by $Y_i = X_1/100$. Since Y_i are *iid*, the distribution of S is approximately normal with mean $\mathbf{E}[S]$ and variance $\text{var}(S)$. Thus, $\mathbf{P}(S > 3) = 1 - \mathbf{P}(S \leq 3) \approx 1 - \Phi\left(\frac{3 - \mathbf{E}(S)}{\sqrt{\text{var}(S)}}\right)$. Now,

$$\begin{aligned}
\mathbf{E}[X_i] &= \int_0^4 x\frac{x}{8}\,dx = \frac{x^3}{24}\bigg|_0^4 = \frac{8}{3} \\
\text{var}(X_i) &= \mathbf{E}[X_i^2] - (\mathbf{E}[X_i])^2 = \int_0^4 x^2\frac{x}{8}\,dx - \left(\frac{8}{3}\right)^2 = \frac{x^4}{32}\bigg|_0^4 - \left(\frac{8}{3}\right)^2 = \frac{8}{9}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{E}[S] &= \frac{1}{100}\mathbf{E}[X_i] + \ldots + \frac{1}{100}\mathbf{E}[X_{100}] = 8/3. \\
\text{var}(S) &= \frac{1}{100^2}\text{var}(X_i) + \ldots + \frac{1}{100^2}\text{var}(X_i) = \frac{8}{9} \times \frac{1}{100}.
\end{aligned}$$

and

$$\mathbf{P}(S > 3) \approx 1 - \Phi\left(\frac{3 - 8/3}{\sqrt{\frac{8}{9} \times \frac{1}{100}}}\right) = 1 - \Phi\left(\frac{5}{\sqrt{2}}\right).$$

i. Let X_i, $i = 1, 2, \ldots$ be independent random variables all distributed according to the PDF $f_X(x) = 1$, $0 \le x \le 1$. Define $Y_n = X_1 X_2 X_3 \ldots X_n$, for some integer n. Then $\text{var}(Y_n)$ is equal to

(i) $\frac{n}{12}$

(ii) $\boxed{\frac{1}{3^n} - \frac{1}{4^n}}$

(iii) $\frac{1}{12^n}$

(iv) $\frac{1}{12}$

Solution: Since $X_1, \ldots, X_n$ are independent, we have that $\mathbf{E}[Y_n] = \mathbf{E}[X_1] \times \ldots \times \mathbf{E}[X_n]$. Similarly, $\mathbf{E}[Y_n^2] = \mathbf{E}[X_1^2] \times \ldots \times \mathbf{E}[X_n^2]$. Since $\mathbf{E}[X_i] = 1/2$ and $\mathbf{E}[X_i^2] = 1/3$ for $i = 1, \ldots, n$, it follows that $\text{var}(S_n) = \mathbf{E}[Y_n^2] - (\mathbf{E}[Y_n])^2 = \frac{1}{3^n} - \frac{1}{4^n}$.

Question 2

Each Mac book has a lifetime that is exponentially distributed with parameter λ. The lifetime of Mac books are independent of each other. Suppose you have two Mac books, which you begin using at the same time. Define T_1 as the time of the first ~~laptop~~ failure and T_2 as the time of the second ~~laptop~~ failure.

a. Compute $f_{T_1}(t_1)$.

Solution

Let M_1 be the life time of mac book 1 and M_2 the lifetime of mac book 2, where M_1 and M_2 are iid exponential random variables with CDF $F_M(m) = 1 - e^{-\lambda m}$. T_1, the time of the first mac book failure, is the minimum of M_1 and M_2. To derive the distribution of T_1, we first find the CDF $F_{T_1}(t)$, and then differentiate to find the PDF $f_{T_1}(t)$.

$$\begin{aligned}
F_{T_1}(t) &= P(\min(M_1, M_2) < t) \\
&= 1 - P(\min(M_1, M_2) \ge t) \\
&= 1 - P(M_1 \ge t)P(M_2 \ge t) \\
&= 1 - (1 - F_M(t))^2 \\
&= 1 - e^{-2\lambda t} \qquad t \ge 0
\end{aligned}$$

Differentiating $F_{T_1}(t)$ with respect to t yields:

$$f_{T_1}(t) = 2\lambda e^{-2\lambda t} \qquad t \ge 0$$

b. Let $X = T_2 - T_1$. Compute $f_{X|T_1}(x|t_1)$.

Solution

Conditioned on the time of the first mac book failure, the time until the other mac book fails is an exponential random variable by the memoryless property. The memoryless property tells us that regardless of the elapsed life time of the mac book, the time until failure has the same exponential CDF. Consequently,

$$f_{X|T_1}(x) = \lambda e^{-\lambda x} \qquad x \geq 0.$$

c. Is X independent of T_1? Give a mathematical justification for your answer.

Solution

Since we have shown in 2(c) that $f_{X|T_1}(x \mid t)$ does not depend on t, X and T_1 are independent.

d. Compute $f_{T_2}(t_2)$ and $\mathbf{E}[T_2]$.

Solution

The time of the second laptop failure T_2 is equal to $T_1 + X$. Since X and T_1 were shown to be independent in 2(b), we convolve the densities found in 2(a) and 2(b) to determine $f_{T_2}(t)$.

$$\begin{aligned}
f_{T_2}(t) &= \int_0^\infty f_{T_1}(\tau) f_X(t-\tau) d\tau \\
&= \int_0^t 2(\lambda)^2 e^{-2\lambda\tau} e^{-\lambda(t-\tau)} d\tau \\
&= 2\lambda e^{-\lambda t} \int_0^t \lambda e^{-\lambda\tau} d\tau \\
&= 2\lambda e^{-\lambda t}(1 - e^{-\lambda t}) \qquad t \geq 0
\end{aligned}$$

Also, by the linearity of expectation, we have that $\mathbf{E}[T_2] = \mathbf{E}[T_1] + \mathbf{E}[X] = \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{2\lambda}$.

An equivalent method for solving this problem is to note that T_2 is the maximum of M_1 and M_2, and deriving the distribution of T_2 in our standard CDF to PDF method:

$$\begin{aligned}
F_{T_2}(t) &= \mathbf{P}(\max(M_1, M_2) < t) \\
&= \mathbf{P}(M_2 \leq t) P(M_2 \leq t) \\
&= F_M(t)^2 \\
&= 1 - 2e^{-\lambda t} + e^{-2\lambda t} \qquad t \geq 0
\end{aligned}$$

Differentiating $F_{T_2}(t)$ with respect to t yields:

$$f_{T_2}(t) = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t} \qquad t \geq 0$$

which is equivalent to our solution by convolution above.

Finally, from the above density we obtain that $\mathbf{E}[T_2] = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}$, which matches our earlier solution.

e. Now suppose you have 100 Mac books, and let Y be the time of the first laptop failure. Find the best answer for $\mathbf{P}(Y < 0.01)$

Solution

Y is equal to the minimum of 100 independent exponential random variables. Following the derivation in (a), we determine by analogy:

$$f_Y(y) = 100\lambda e^{-100\lambda y} \qquad t \geq 0$$

Integrating over y from 0 to .01, we find $\mathbf{P}(Y < .01) = 1 - e^{-\lambda}$.

Your friend, Charlie, loves Mac books so much he buys S new Mac books every day! On any given day S is equally likely to be 4 or 8, and all days are independent from each other. Let S_{100} be the number of Mac books Charlie buys over the next 100 days.

f. (6 pts) Find the best approximation for $\mathbf{P}(S_{100} \leq 608)$. Express your final answer in terms of $\Phi(\cdot)$, the CDF of the standard normal.

Solution

Using the De Moivre - Laplace Approximation to the Binomial, and noting that the step size between values that S can take on is 4,

$$\begin{aligned} P(S_{100} \leq 608) &= \Phi\left(\frac{608 + 2 - 100 \times 6}{\sqrt{100 \times 4}}\right) \\ &= \Phi\left(\frac{10}{20}\right) \\ &= \Phi\left(.5\right) \end{aligned}$$

Question 3

Saif is a well intentioned though slightly indecisive fellow. Every morning he flips a coin to decide where to go. If the coin is heads he drives to the mall, if it comes up tails he volunteers at the local shelter. Saif's coin is not necessarily fair, rather it possesses a probability of heads equal to q. We do not know q, but we do know it is well-modeled by a random variable Q where the density of Q is

$$f_Q(q) = \begin{cases} 2q & \text{for } \ 0 \leq q \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume conditioned on Q each coin flip is independent. Note parts a, b, c, and $\{d, e\}$ may be answered independent of each other.

a. (4 pts) What's the probability that Saif goes to the local shelter if he flips the coin once?

Solution

Let X_i be the outcome of a coin toss on the i^{th} trial, where $X_i = 1$ if the coin lands 'heads', and $X_i = 0$ if the coin lands 'tails.' By the total probability theorem:

$$\begin{aligned}
\mathbf{P}(X_1 = 0) &= \int_0^1 P(X_1 = 0 \mid Q = q) f(q) dq \\
&= \int_0^1 (1-q) 2q \, dq \\
&= \frac{1}{3}
\end{aligned}$$

In an attempt to promote virtuous behavior, Saif's father offers to pay him \$4 every day he volunteers at the local shelter. Define X as Saif's payout if he flips the coin every morning for the next 30 days.

b. Find $\text{var}(X)$

Solution Let Y_i be a Bernoulli random variable describing the outcome of a coin tossed on morning i. Then, $Y_i = 1$ corresponds to the event that on morning i, Saif goes to the local shelter; $Y_i = 0$ corresponds to the event that on morning i, Saif goes to the mall. Assuming that the coin lands heads with probability q, i.e. that $Q = q$, we have that $P(Y_i = 1) = q$, and $P(Y_i = 0) = 1 - q$ for $i = 1, \ldots, 30$.

Saif's payout for next 30 days is described by random variable $X = 4(Y_1 + Y_2 + \cdots + Y_{30})$.

$$\begin{aligned}
\text{var}(X) &= 16 \, \text{var}(Y_1 + Y_2 + \cdots + Y_{30}) \\
&= 16 \, \text{var}(\mathbf{E}[Y_1 + Y_2 + \cdots + Y_{30} \mid Q]) + \mathbf{E}[\text{var}(Y_1 + Y_2 + \cdots + Y_{30} \mid Q)]
\end{aligned}$$

Now note that, conditioned on $Q = q$, $Y_1, \ldots, Y_{30}$ are independent. Thus, $\text{var}(Y_1 + Y_2 + \cdots + Y_{30} \mid Q) = \text{var}(Y_1 \mid Q) + \ldots + \text{var}(Y_{30} \mid Q)$. So,

$$\begin{aligned}
\text{var}(X) &= 16 \, \text{var}(30Q) + 16 \, \mathbf{E}[\text{var}(Y_1 \mid Q) + \ldots + \text{var}(Y_{30} \mid Q)] \\
&= 16 \times 30^2 \text{var}(Q) + 16 \times 30 \, \mathbf{E}[Q(1-Q)] \\
&= 16 \times 30^2 (\mathbf{E}[Q^2] - (\mathbf{E}[Q])^2) + 16 \times 30 (\mathbf{E}[Q] - \mathbf{E}[Q^2]) \\
&= 16 \times 30^2 (1/2 - 4/9) + 16 \times 30 (2/3 - 1/2) = 880
\end{aligned}$$

since $\mathbf{E}[Q] = \int_0^1 2q^2 \, dq = 2/3$ and $\mathbf{E}[Q^2] = \int_0^1 2q^3 \, dq = 1/2$.

Let event B be that Saif goes to the local shelter at least once in k days.

c. Find the conditional density of Q given B, $f_{Q|B}(q)$

Solution
By Bayes Rule:

$$\begin{aligned} f_{Q|B}(q) &= \frac{P(B \mid Q = q) f_Q(q)}{\int P(B \mid Q = q) f_Q(q) dq} \\ &= \frac{(1 - q^k) 2q}{\int_0^1 (1 - q^k) 2q \, dq} \\ &= \frac{2q(1 - q^k)}{1 - 2/(k+2)} \qquad 0 \le q \le 1 \end{aligned}$$

While shopping at the mall, Saif gets a call from his sister Mais. They agree to meet at the Coco Cabana Court yard at exactly 1:30PM. Unfortunately Mais arrives Z minutes late, where Z is a continuous uniform random variable from zero to 10 minutes. Saif is furious that Mais has kept him waiting, and demands Mais pay him R dollars, where $R = \exp(Z + 2)$.

e. Find Saif's expected payout, $\mathbf{E}[R]$.

Solution

$$\begin{aligned} \mathbf{E}[R] &= \int_0^{10} e^{z+2} f(z) dz \\ &= \frac{e^2}{10} \int_0^{10} e^z dz \\ &= \frac{e^{12} - e^2}{10} \end{aligned}$$

f. Find the density of Saif's payout, $f_R(r)$.

Solution

$$\begin{aligned} F_R(r) &= \mathbf{P}(e^{Z+2} \le r) \\ &= \mathbf{P}(Z + 2 \le \ln(r)) \\ &= \int_0^{\ln(r)-2} \frac{1}{10} dz \\ &= \frac{\ln(r) - 2}{10} \qquad e^2 \le r \le e^{12} \end{aligned}$$

$$f_R(r) = \frac{1}{10r} \qquad e^2 \le r \le e^{12}$$