| | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |

**The standard normal table.** The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \leq y)$, where $Y$ is a standard normal random variable, for $y$ between 0 and 1.99. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When $y$ is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

**Problem 2.** (42 points)
The random variable $X$ is exponential with parameter 1. Given the value $x$ of $X$, the random variable $Y$ is exponential with parameter equal to $x$ (and mean $1/x$).

*Note:* Some useful integrals, for $\lambda > 0$:

$$\int_0^\infty xe^{-\lambda x}\,dx = \frac{1}{\lambda^2}, \qquad \int_0^\infty x^2 e^{-\lambda x}\,dx = \frac{2}{\lambda^3}.$$

(a) (7 points) Find the joint PDF of $X$ and $Y$.

(b) (7 points) Find the marginal PDF of $Y$.

(c) (7 points) Find the conditional PDF of $X$, given that $Y = 2$.

(d) (7 points) Find the conditional expectation of $X$, given that $Y = 2$.

(e) (7 points) Find the conditional PDF of $Y$, given that $X = 2$ and $Y \geq 3$.

(f) (7 points) Find the PDF of $e^{2X}$.

**Problem 3.** (10 points)
For the following questions, mark the correct answer. If you get it right, you receive 5 points for that question. You receive no credit if you get it wrong. A justification is not required and will not be taken into account.

Let $X$ and $Y$ be continuous random variables. Let $N$ be a discrete random variable.

(a) (5 points) The quantity $\mathbf{E}[X \mid Y]$ is always:

(i) A number.
(ii) A discrete random variable.
(iii) A continuous random variable.
(iv) Not enough information to choose between (i)-(iii).

(b) (5 points) The quantity $\mathbf{E}[\,\mathbf{E}[X \mid Y, N] \mid N]$ is always:

(i) A number.
(ii) A discrete random variable.
(iii) A continuous random variable.
(iv) Not enough information to choose between (i)-(iii).

**Problem 4.** (25 points)
The probability of obtaining heads in a single flip of a certain coin is itself a random variable, denoted by $Q$, which is uniformly distributed in $[0, 1]$. Let $X = 1$ if the coin flip results in heads, and $X = 0$ if the coin flip results in tails.

(a) (i) (5 points) Find the mean of $X$.
(ii) (5 points) Find the variance of $X$.

(b) (7 points) Find the covariance of $X$ and $Q$.

(c) (8 points) Find the conditional PDF of $Q$ given that $X = 1$.

**Problem 5.** (21 points)
Let $X$ and $Y$ be **independent continuous** random variables with marginal PDFs $f_X$ and $f_Y$, and marginal CDFs $F_X$ and $F_Y$, respectively. Let

$$S = \min\{X, Y\}, \qquad L = \max\{X, Y\}.$$

(a) (7 points) If $X$ and $Y$ are standard normal, find the probability that $S \geq 1$.

(b) (7 points) Fix some $s$ and $\ell$ with $s \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \text{ and } L \leq \ell)$$

involving $F_X$ and $F_Y$, and no integrals.

(c) (7 points) Assume that $s \leq s + \delta \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \leq s + \delta,\ \ell \leq L \leq \ell + \delta),$$

as an integral involving $f_X$ and $f_Y$.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

**Problem 2.** (42 points)
The random variable $X$ is exponential with parameter 1. Given the value $x$ of $X$, the random variable $Y$ is exponential with parameter equal to $x$ (and mean $1/x$).

*Note:* Some useful integrals, for $\lambda > 0$:

$$\int_0^\infty xe^{-\lambda x}\,dx = \frac{1}{\lambda^2}, \qquad \int_0^\infty x^2 e^{-\lambda x}\,dx = \frac{2}{\lambda^3}.$$

(a) (7 points) Find the joint PDF of $X$ and $Y$.

(b) (7 points) Find the marginal PDF of $Y$.

(c) (7 points) Find the conditional PDF of $X$, given that $Y = 2$.

(d) (7 points) Find the conditional expectation of $X$, given that $Y = 2$.

(e) (7 points) Find the conditional PDF of $Y$, given that $X = 2$ and $Y \geq 3$.

(f) (7 points) Find the PDF of $e^{2X}$.

**Problem 3.** (10 points)
For the following questions, mark the correct answer. If you get it right, you receive 5 points for that question. You receive no credit if you get it wrong. A justification is not required and will not be taken into account.

Let $X$ and $Y$ be continuous random variables. Let $N$ be a discrete random variable.

(a) (5 points) The quantity $\mathbf{E}[X \mid Y]$ is always:

(i) A number.

(ii) A discrete random variable.

(iii) A continuous random variable.

(iv) Not enough information to choose between (i)-(iii).

(b) (5 points) The quantity $\mathbf{E}[\,\mathbf{E}[X \mid Y, N] \mid N]$ is always:

(i) A number.

(ii) A discrete random variable.

(iii) A continuous random variable.

(iv) Not enough information to choose between (i)-(iii).

**Problem 4.** (25 points)
The probability of obtaining heads in a single flip of a certain coin is itself a random variable, denoted by $Q$, which is uniformly distributed in $[0, 1]$. Let $X = 1$ if the coin flip results in heads, and $X = 0$ if the coin flip results in tails.

(a) (i) (5 points) Find the mean of $X$.

(ii) (5 points) Find the variance of $X$.

(b) (7 points) Find the covariance of $X$ and $Q$.

(c) (8 points) Find the conditional PDF of $Q$ given that $X = 1$.

**Problem 5.** (21 points)
Let $X$ and $Y$ be **independent continuous** random variables with marginal PDFs $f_X$ and $f_Y$, and marginal CDFs $F_X$ and $F_Y$, respectively. Let

$$S = \min\{X, Y\}, \qquad L = \max\{X, Y\}.$$

(a) (7 points) If $X$ and $Y$ are standard normal, find the probability that $S \geq 1$.

(b) (7 points) Fix some $s$ and $\ell$ with $s \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \text{ and } L \leq \ell)$$

involving $F_X$ and $F_Y$, and no integrals.

(c) (7 points) Assume that $s \leq s + \delta \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \leq s + \delta,\ \ell \leq L \leq \ell + \delta),$$

as an integral involving $f_X$ and $f_Y$.