Solutions to Quiz 1

Problem 1 (25 points). Consider the following system specifications[1].

1. The system is in multiuser state iff it is operating normally.
2. If the system is operating normally, then the kernel is functioning.
3. The kernel is not functioning or the system is in interrupt mode.
4. If the system is not in multiuser state, then it is in interrupt mode.
5. The system is not in interrupt mode.

(a) (0 points) To make sense of these confusing conditions, let's introduce four Boolean variables.

$$M ::= \text{in Multiuser state} \tag{1}$$
$$N ::= \text{operating Normally} \tag{2}$$
$$K ::= \text{Kernel is functioning} \tag{3}$$
$$I ::= \text{in Interrupt mode} \tag{4}$$

Translate the five statements in the specification into propositional logic notation: $\wedge, \vee, \neg, \longrightarrow, \longleftrightarrow$

Solution.

$$M \longleftrightarrow N \tag{5}$$
$$N \longrightarrow K \tag{6}$$
$$\neg K \quad \vee \quad I \tag{7}$$
$$\neg M \longrightarrow I \tag{8}$$
$$\neg I \tag{9}$$

■

[1]Rosen, Exercise 1.1.35

(b) (0 points) Are these system specifications consistent? _____. Prove it!

Solution. There are several ways to approach this problem. One is to construct a truth table with sixteen lines—one for each way of assigning truth values to the four variables M, N, K, and I.

We can avoid the cumbersome truthtable if we reason by cases. Case 1: I is true. Then the last formula (9) is false, and the whole specification is false. Case 2: I is false. Now formula (8) can be true only if $\neg M$ is false, that is, only if M is true. Likewise, formula (7) can be true only if $\neg K$ is true, that is, K is false.

Since K is false, formula (6) can be true only if N is false. Thus, we have deduced that in order to be consistent in this case, we must have

$$\begin{aligned} I &= \text{false} \\ M &= \text{true} \\ K &= \text{false} \\ N &= \text{false.} \end{aligned}$$

But now formula (5) is false, so it is impossible for all the formulas to be true: the system is inconsistent. ■

Problem 2 (20 points). For each of the following logical formulas with domain of discourse the natural numbers, $\mathbb{N}$, indicate whether it is a possible formulation of

I: the **I**nduction Axiom,

S: the **S**trong Induction Axiom,

L: the **L**east Number Principle (also known as Well-ordering), or

N: **N**one of these.

For example, the ordinary Induction Axiom could be expressed by the following formula, so it gets labelled "I".

$$(P(0) \wedge [\forall k\ P(k) \longrightarrow P(k+1)]) \longrightarrow \forall k\ P(k) \qquad \underline{\ \text{I}\ }$$

This is a multiple choice problem: do not explain your answer.

(a) (0 points) $(P(b) \wedge [\forall k \geq b\ P(k) \longrightarrow P(k+1)]) \longrightarrow \forall k \geq b\ P(k)$ ____

Solution. I. This is a perfect formulation of the Induction Axiom. b is used for the base case; $P(k) \longrightarrow P(k+1)$ is the inductive case. ■

(b) (0 points) $(P(b) \wedge [\forall k \leq b\ P(k) \longrightarrow P(k+1)]) \longrightarrow \forall k \leq b\ P(k)$ ____

Solution. N. The two occurences of $k <= b$ should have been $k >= b$ ■

(c) (0 points) $[\forall b\ (\forall k < b\ P(k)) \longrightarrow P(b)] \longrightarrow \forall k\ P(k)$ ____

Solution. S. Since you are assuming $P(k)$ for all $k < b$, this is strong induction. ■

(d) (0 points) $(\exists n\ P(n)) \longrightarrow \exists n\ \forall k < n\ \overline{P(k)}$ ____

Solution. N. This statement is in fact *always* true; when $n = 0$, $\forall k < n\ \overline{P(k)}$. It should say $P(n) \wedge \forall k < n; \overline{P(k)}$ ■

(e) (0 points) $\forall n\ [P(n) \longrightarrow (\exists n\ P(n) \wedge \forall k < n\ \overline{P(k)})]$ ____

Solution. L. This is a valid formulation; it does not have that problem in (2d). ■

Problem 3 (20 points). Classify each of the following binary relations as

E: An equivalence relation.

T: A Total order,

P: A Partial order that is not total.

S: A Symmetric relation that is not transitive.

N: None of the above.

This is a multiple choice problem: do not explain your answer.

(a) (0 points) The relation xRy between times of day such that x and y are at most twenty minutes apart. ____

Solution. **S**: This relation is reflexive and symmetric. It is not transitive; Consider the counter-example: 1:00$R$1:15, 1:15$R$1:22 but $\neg$1:00$R$1:22. ■

(b) (0 points) The relation xRy between times of day such that x is more than twenty minutes later than y. ____

Solution. **P**: This relation is antisymmetric and transitive but not reflexive (since a time isn't 20 minutes after itself). This is not a total ordering because some times are incomparable to each other. For example, 1:15 is incomparable to 1:22.

Note: This answer assumes that the question was referring to the moments in time in a *single* day. Otherwise, one could argue that 1:00 precedes 1:22 on tuesday, but 1:22 on tuesday precedes 1:00 on wednesday, and then the relation would not be antisymmetric.■

(c) (0 points) The relation xRy over all words in this sentence such that x does not appear after y. (Consider "x", "y", and "xRy" to be words.) ____

Solution. **T**: Because all the words in the sentence are unique, this relation is transitive, antisymmetric, and reflexive. This makes the relation a partial order. However, the relation is also a total order, because any two elements are comparable. ■

(d) (0 points) The relation xRy over all words in this sentence such that word x does not appear after word y. ____

Solution. **P**: This is the same as saying that *all* appearances of a word x occur before all appearances of a word y. While apparently very similar to part **c**, this relation is not reflexive because the word word appears twice in the sentence. It is transitive and antisymmetric, but not a total order because all the words between the two occurrences of word are incomparable to word. ■

(e) (0 points) The relation xRy over all words in this sentence such that the final appearance of y occurs after x. ____

Solution. **N**: This relation is not reflexive and transitive. It is neither symmetric nor antisymmetric; the word the occurs twice, all the others once. All words w, between the two occurrences of the satisfy $w R$ the and theRw. So R is not anti-symmetric. But "x" is maximal, so it's not symmetric either. ■

Problem 4 (35 points). To encourage collaborative study, the 6.042 staff is considering assigning each student to a study group with two or three other students. Prove that as long as the enrollment is large enough, the class can always be divided into such study groups.

Solution. *Proof.* The proof is by strong induction. The induction hypothesis is that a class with $n \geq 6$ students can be divided into teams of 3 or 4. More precisely

$$P(n) ::= \quad n \geq 6 \longrightarrow \exists x, y \in \mathbb{N}\ 3x + 4y = n.$$

For any $n \geq 0$, we may assume $P(6), \ldots, P(n-1)$ to prove $P(n)$.

Case 1: ($n < 6$). $P(n)$ holds because the hypothesis $n \geq 6$ is false.

Case 2: ($n = 6, 7,$ or 8) $P(6)$ is true because there could be two teams of 3, $P(7)$ is true because there could be a team of 3 and a team of 4, and $P(8)$ is true because there could be two teams of 4.

Case 3: ($n \geq 9$). Of course $n > n-3$ so $P(n-3)$ holds by the strong induction hypothesis. But $n - 3 \geq 6$, so $P(n-3)$ implies $3x' + 4y' = n - 3$ for some $x', y' \in \mathbb{N}$, and therefore $3x + 4y = n$ where $x ::= x' + 1$ and $y ::= y'$. So $P(n)$ holds, as required. $\square$

$\blacksquare$