Mathematics for Computer Science
MIT 6.042J/18.062J

Propositional Logic, II

Proof by Contradiction
Proof by Cases

Proof by Contradiction

$$\frac{\neg P \rightarrow (Q \wedge \neg Q)}{P}$$

Alternatively, $$\frac{\bar{P} \rightarrow \text{F}}{P}$$

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

- Suppose $\sqrt{2}$ was rational.
- Choose m, n integers without common prime factors (always possible) such that

$$\sqrt{2} = \frac{m}{n}$$

- Show that m & n are both even, a **contradiction**!

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

$$\sqrt{2} = \frac{m}{n}$$

$$\sqrt{2}\,n = m$$

$$2n^2 = m^2$$

so m is even.

so can assume $m = 2l$

$$m^2 = 4l^2$$

$$2n^2 = 4l^2$$

$$n^2 = 2l^2$$

so n is even.

Quickie

Proof assumes that

If m^2 is even, then m is even.

Why!

Team Problem

Problem 1

Proof by Truth Tables

DeMorgan's Law

$\neg(P \vee Q)$ is *equivalent* to $\bar{P} \wedge \bar{Q}$

P	Q	$\neg(P \vee Q)$	
T	T	F	T
T	F	F	T
F	T	F	T
F	F	T	F

$\bar{P}$	$\bar{Q}$	$\bar{P} \wedge \bar{Q}$
F	F	F
F	T	F
T	F	F
T	T	T

Proof by Deductions

A student is trying to prove that propositions P, Q, and R are all true. She proceeds as follows.
First, she proves three facts:

- P implies Q
- Q implies R
- R implies P.

Then she concludes,
``Thus P, Q, and R are obviously all true."

Proposed Deduction Rule

From: P implies Q, Q implies R, R implies P
Conclude: P, Q, and R are true.

$$\frac{(P \to Q),\ (Q \to R),\ (R \to P)}{P \wedge Q \wedge R}$$

Sound Rule?

Conclusion true whenever all antecedents true.

$P \to Q \quad Q \to R \quad R \to P \qquad P \wedge Q \wedge R$

Antecedents *Conclusion*

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

$P \to Q \quad Q \to R \quad R \to P \qquad P \wedge Q \wedge R$

Antecedents *Conclusion*

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R	$p \to q$	$q \to r$	$r \to p$	$p \wedge q \wedge r$	sound?
T	T	T	T	T	T	T	
T	T	F	T	F	T	F	
T	F	T	F	T	T	F	
T	F	F	F	T	T	F	
F	T	T	T	T	F	F	
F	T	F	T	F	T	F	
F	F	T	T	T	F	F	
F	F	F	T	T	T	F	

Antecedents *Conclusion*

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R	$p \to q$	$q \to r$	$r \to p$	$p \wedge q \wedge r$	sound?
T	T	T	T	T	T	T	
T	T	F	T	F	T	F	OK
T	F	T	F	T	T	F	OK
T	F	F	F	T	T	F	OK
F	T	T	T	T	F	F	OK
F	T	F	T	F	T	F	OK
F	F	T	T	T	F	F	OK
F	F	F	T	T	T	F	

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R	$p \to q$	$q \to r$	$r \to p$	$p \wedge q \wedge r$	sound?
T	T	T	T	T	T	T	OK
T	T	F	T	F	T	F	OK
T	F	T	F	T	T	F	OK
T	F	F	F	T	T	F	OK
F	T	T	T	T	F	F	OK
F	T	F	T	F	T	F	OK
F	F	T	T	T	F	F	OK
F	F	F	T	T	T	F	

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R	$p \to q$	$q \to r$	$r \to p$	$p \wedge q \wedge r$	sound?
T	T	T	T	T	T	T	OK
T	T	F	T	F	T	F	OK
T	F	T	F	T	T	F	OK
T	F	F	F	T	T	F	OK
F	T	T	T	T	F	F	OK
F	T	F	T	F	T	F	OK
F	F	T	T	T	F	F	OK
F	F	F	T	T	T	F	

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R	$p \to q$	$q \to r$	$r \to p$	$p \wedge q \wedge r$	sound?
T	T	T	T	T	T	T	OK
T	T	F	T	F	T	F	OK
T	F	T	F	T	T	F	OK
T	F	F	F	T	T	F	OK
F	T	T	T	T	F	F	OK
F	T	F	T	F	T	F	OK
F	F	T	T	T	F	F	OK
F	F	F	T	T	T	F	NOT OK!

Reasoning by Cases

Quicker proof of unsoundness than from truth tables

Quicker by Cases

$$\frac{P \to Q,\ \ Q \to R,\ \ R \to P}{P \wedge Q \wedge R}$$

Case 1: P is true. Now, if antecedents are true, then Q must be true (because P implies Q). Then R must be true (because Q implies R). So the conclusion $P \wedge Q \wedge R$ is true. This case is OK.

Quicker by Cases

$$\frac{P \to Q,\ Q \to R,\ R \to P}{P \wedge Q \wedge R}$$

Case 2: P is false. To make antecedents true,
R must be false (because R implies P), so
Q must be false (because Q implies R).
This assignment does make the antecedents true,
but the conclusion $P \wedge Q \wedge R$ is (very) False.

This case is not OK.

Goldbach Conjecture

Every even integer greater than 2 is the sum of two primes.

Evidence:
$$4 = 2 + 2$$
$$6 = 3 + 3$$
$$8 = 5 + 3$$
$$\vdots$$
$$20 = ?\ \ 13 + 7$$

Goldbach Conjecture

True for all even numbers with
up to 13 digits! (Rosen, p.182)

It remains an OPEN problem:
no counterexample, no proof.

UNTIL NOW!…

Goldbach Conjecture

The answer is on my desk!

(Proof by Cases)

Team Problem

Problems 2 & 3