PROFESSOR: In 6.042, we're going to be pretty concerned with proofs. We're going to try to help you learn how to do rudimentary proofs and not be afraid of them. The most important skill, in some ways, is the ability to distinguish a very plausible argument that might not be totally right from a proof which is totally right. That's an important skill.

And it's a basic understanding of what math is. It's that distinction between knowing when a thing is mathematically, absolutely unarguable and inevitable as opposed to something that's just very likely. It's interesting. Physicists by and large do a lot of math, and they tend not to worry so much about proofs. But all the theoreticians and the mathematicians are in agreement that you don't really understand the subject until you know how to prove the basic facts.

Pragmatically, the value of proofs is that there's an awful lot of content in this subject and in many other mathematical subjects. And if your only way to figure out what the exact details are is memorization, you're going to get lost. Most of these rules and theorems that we prove, I can never remember them exactly. But I know how to prove them, so I can debug them and get them exactly right.

So let's begin by looking at just examples of proofs before we start to try to get abstract about what they are. And we'll look at a famous theorem that you've all seen from early on in high school, the Pythagorean theorem. It says that if I have a right triangle with sides a and b and hypotenuse c, then there's a relationship between a, b and c-- namely that a squared plus b squared equals c squared.

Now, this is, as I said, completely familiar. But is it obvious? Well, every once in a while, students say it's obvious, but what I think they really mean is that it's familiar. It's not obvious.

Part of the argument for the fact that it's not obvious is that for thousands of years, people have kept feeling the need to prove it in order to be sure that it's true and explain why it's true. There's a citation in the notes of a website devoted to collecting Pythagorean theorem proofs. There's over a hundred of them, including one by a former president of the United States.

So let's look at one of my favorite proofs of the Pythagorean theorem. And it goes this way. There are four triangles that are all the same size, four copies of this abc triangle, which we've put in different colors to distinguish them, and a square, which for the moment, is of unknown size.

And the proof of the Pythagorean theorem is going to consist of taking these four shapes and reassembling them so that they form a c by c square first, and then finding a second arrangement so that they form two squares-- an a by a square and a b by b square. Then by the theorem of conservation of paper or conservation of area, it has to be that the c by c area is the same as the a by a plus b by b area. And so a squared plus b squared is equal to c squared.

Well, let's look at those rearrangements. And probably, you should take a moment to try it yourself before I pop the solution up. But there's the solution to the first one. It's the easier of the two.

This is the c by c arrangement. The hint is that if it's going to be c by c, you don't have a lot of choice except to put the c [? long ?] hypotenuses on the outside. And then it's a matter of just fiddling the triangles around so they fit together. And you discover there's a square in the middle. And that's just where that extra square that is provided will fit.

Also, this enables you to figure out what the dimensions of the square are. Because if you look at it, this is a b side. We're letting b be the longer of the two sides of the triangle. And this is the a side, the shorter side of another triangle. So what's left here has to be b minus a. So now we know that it's a b minus a by b minus a square from this arrangement. And that's what we've indicated here.

Now, the next arrangement is the following. We're going to take two of the triangles and form a rectangle, another two triangles to from a rectangle, line them up in this way, and fit the b minus a by b minus a square there. Now, where are the two squares? Well, I didn't say that the a by a and a and b by b square needed to be separate. In fact, they're not. They're attached. But where are they?

Well, let's look at this line. How long is it? Well, it's a plus b minus a long, which means that it's b long. And suddenly, there is a b and there's a b, and I've got a b by b rectangle right there.

But wait a second. Here's a b minus a, and it's lined up against a b side. So if I look at what's left, it's b minus b minus a. It tells me that that little piece is a. And so sure enough, when I add this hidden line-- conceptual line to separate the two squares, this part's a by a, and that part's b by b. And we've proved the Pythagorean theorem.

So what about this process? It's really very elegant, and it's absolutely correct. And I hope it's kind of convincing. And so this is a wonderful case of a proof by picture that really works in this case.

But unfortunately, proofs by pictures worry mathematicians, and they're illegitimately worrisome because there's lots of hidden assumptions. An exercise that you can go through is to go back and think about all of the geometric information that's kind of being taken for granted in this picture. Like over here, how did we know that that was a right angle, that this thing was a rectangle? We needed that to be a right angle because we were claiming that this was a square. Well, how did we know that that was a rectangle?

Well, the answers are obvious. We're using the fact that the complementary angles of a right triangle sum to 90 degrees because the angles of a triangle in general sum to 180 degrees. We're using that in a bunch of other places.

We're also using the fact that this is a straight line, which may or may not be obvious. But it's true, and that's why it's safe to add those distances to figure out what it was. My point is that there are really a whole lot of hidden assumptions in the diagram that it's easy to overlook and be fooled by.

So let me show you an example of getting fooled by a proof by diagram. And here is how to get infinitely rich. Let's imagine that I have a 10 by 11 piece of gold foil. Actually, they could be slabs of gold, but let's think of this as a rectangular shape that's made out of gold. And it's going to be rectangles. Those are right angles there.

And what I'm going to do is mark off the corners. I'm going to mark off a length down of 1, and then I'm going to mark off a length of 1 and shift it so that it touches the diagonal, and do the same thing in this lower corner. And now, let's just shift these shapes, the top one going southwest and the second one going northeast.

And what I wind up with is this picture so that I've now got those little red triangles protruding above the shape. OK, cool. Well, what do we know? This is now side 10 because I've subtracted 1 from its length here . And this is side 11 because that used to be 10, and I've added 1 to its length there.

So that's cool because now, what I can do is take those protruding triangles out, and they'll form a little 1 by 1 square. And suddenly, I have this little bit of gold that's extra.

But look what's here. It's a 10 by 11 rectangular shape of gold foil again. So I just rotate this by 90 degrees, and I start all over again. I can keep generating these little 1 by 1 shapes of gold foil forever. I could get infinitely rich.

OK, well there's something wrong with that. It's violating all kinds of conservation principles, not to mention that it would undermine the price of gold. So what's the bug? Well, you probably can spot this, but maybe you've been fooled. What's going on is there's an implicit assumption that those little triangles that I cut off were right triangles and that this line that I claimed was of length 11 was a straight line, and it's not.

Those triangles have two sides that are of length 1. They're isosceles triangles, but they're lying up against a diagonal that's not 45 degrees. And so they're not right triangles. And that line isn't straight. And 10 and 11 were close enough that it wasn't visually obvious.

So this is a way to simply put one over on you with a proof by picture. And if I had been asked to justify how do I know it's a straight line, that bug would have emerged. But you're not likely to notice that if it isn't visually obvious, which is why we worry about that some of these proofs [? by picture. ?]