Practice Final Exam

- Do not open this quiz booklet until you are directed to do so. Read all the instructions first.
- The quiz contains 6 problems, several with multiple parts. You have 180 minutes to earn 120 points.
- This quiz booklet contains 15 pages, including this one, and a sheet of scratch paper which can be detached.
- This quiz is closed book. You may use two double sided Letter ($8\frac{1}{2}'' \times 11''$) or A4 crib sheets. No calculators or programmable devices are permitted. Cell phones must be put away.
- Write your solutions in the space provided. If you run out of space, continue your answer on the back of the same sheet and make a notation on the front of the sheet.
- Do not waste time deriving facts that we have studied. It is sufficient to cite known results.
- Do not spend too much time on any one problem. Generally, a problem's point value is an indication of how many minutes to spend on it.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Please be neat.
- Good luck!

Problem	Title	Points	Parts	Grade	Initials
0	Name	1	15		
1	True or False	44	11		
2	P, NP & Friends	10	1		
3	Taming MAX-CUT	10	3		
4	Spy Games	10	2		
5	Lots of Spanning trees	25	5		
6	Traveling with the salesman	20	3		
Total		120			

Name: ______________________________

Problem 0. Name. [1 point] Write your name on every page of this exam booklet! Don't forget the cover.

Problem 1. True or False. [44 points] (11 parts)

Circle **T** or **F** for each of the following statements to indicate whether the statement is true or false, respectively, and briefly explain why. Your justification is worth more points than your true-or-false designation.

(a) **T F** [4 points] If problem A can be reduced to 3SAT via a deterministic polynomial-time reduction, and $A \in$ NP, then A is NP-complete.

(b) **T F** [4 points] Let $G = (V, E)$ be a flow network, i.e., a weighted directed graph with a distinguished source vertex s, a sink vertex t, and non-negative capacity $c(u, v)$ for every edge (u, v) in E. Suppose you find an s-t cut C which has edges $e_1, e_2, \ldots, e_k$ and a capacity f. Suppose the value of the maximum s-t flow in G is f.

Now let H be the flow network obtained by adding 1 to the capacity of each edge in C. Then the value of the maximum s-t flow in H is $f + k$.

(c) T F [4 points] Let A and B be optimization problems where it is known that A reduces to B in polynomial time. Additionally, it is known that there exists a polynomial-time 2-approximation for B. Then there must exist a polynomial-time 2-approximation for A.

(d) T F [4 points] There exists a polynomial-time 2-approximation algorithm for the general Traveling Salesman Problem.

(e) T F [4 points] If we use a max-queue instead of a min-queue in Kruskal's MST algorithm, it will return the spanning tree of maximum total cost (instead of returning the spanning tree of minimum total cost). (Assume the input is a weighted connected undirected graph.)

(f) T F [4 points] A randomized algorithm for a decision problem with one-sided-error and correctness probability $1/3$ (that is, if the answer is `YES`, it will always output `YES`, while if the answer is `NO`, it will output `NO` with probability $1/3$) can always be amplified to a correctness probability of 99%.

(g) T F [4 points] Suppose that a randomized algorithm A has expected running time $\Theta(n^2)$ on any input of size n. Then it is possible for some execution of A to take $\Omega(3^n)$ time.

(h) T F [4 points] Building a heap on n elements takes $\Theta(n \lg n)$ time.

(i) T F [4 points] We can evaluate a polynomial of degree-bound n at any set of n points in $O(n \lg n)$ time.

(j) T F [4 points] Suppose that you have two deterministic online algorithms, A_1 and A_2, with a competitive ratios c_1 and c_2 respectively. Consider the randomized algorithm A^* that flips a fair coin once at the beginning; if the coin comes up heads, it runs A_1 from then on; if the coin comes up tails, it runs A_2 from then on. Then the expected competitive ratio of A^* is at least $min\{c_1, c_2\}$.

Problem 2. Taming Max-Cut [10 points] A CUT, in a graph $G = (V, E)$, is a partition of V into two non-intersecting sets A, B. An edge is said to be in the cut if one of its end points is in A and the other is in B. In the MAX-CUT problem, the objective is to maximize the number of edges in the cut. We intend to design an approximation scheme for MAX-CUT. Consider the following scheme. Every vertex $v \in V$ is assigned to A, B uniformly at random.

(a) What is the probability that $e \in E$ is in the cut?

(b) What is the expected number of edges in the cut?

(c) Conclude that the randomized scheme presented above is a 2-approximation to the MAX-CUT.

Problem 3. Lots of Spanning Trees. (5 parts) [25 points] Let $G = (V, E)$ be a connected undirected graph with edge-weight function $w : E \to \mathbb{R}$. Let $w_{\min}$ and $w_{\max}$ denote the minimum and maximum weights, respectively, of the edges in the graph. Do not assume that the edge weights in G are distinct or nonnegative. The following statements may or may not be correct. In each case, either prove the statement is correct or give a counterexample if it is incorrect.

(a) If the graph G has more than $|V| - 1$ edges and there is a unique edge having the largest weight $w_{\max}$, then this edge cannot be part of any minimum spanning tree.

(b) Any edge e with weight w_{min}, must be part of some MST.

(c) If G has a cycle and there is unique edge e which has the minimum weight on this cycle, then e must be part of every MST.

(d) If the edge e is not part of any MST of G, then it must be the maximum weight edge on some cycle in G.

(e) Suppose the edge weights are nonnegative. Then the shortest path between two vertices must be part of some MST.

Problem 4. Traveling with the salesman. [20 points] In the **traveling-salesman problem**, a salesman must visit n cities. Modeling the problem as a complete graph on n vertices, we can say that the salesman wishes to make a **tour** or a hamiltonian cycle, visiting each city exactly only once and finishing at the city he starts from. The salesman incurs a nonnegative integer cost $c(i, j)$ to travel from city i to city j, and the salesman wishes to make a tour whose total cost is minimum, where the total cost is the sum of the individual costs along the edges of the tour.

(a) Formulate the traveling salesman problem as a language.

TSP $=$

(b) Prove that TSP $\in \mathsf{NP}$.

A **hamiltonian cycle** in a graph is a cycle that visits every vertex exactly once. Define the language HAM-CYCLE $= \{< G >$: there is a hamiltonian cycle in $G\}$.

(c) Assuming that HAM-CYCLE is complete for the class NP, prove that TSP is NP-Complete.

SCRATCH PAPER