Lecture 24: Cache-oblivious algorithms II

- Search
  - binary
  - B-ary
  - cache-oblivious
- Sorting
  - mergesorts
  - cache-oblivious

Why LRU block replacement strategy?

$LRU_M \leq 2 \cdot OPT_{M/2}$ [Sleater and Tarjan 1985]

Proof.

- partition block access sequence into maximal phases of M/B distinct blocks
- LRU spends $\leq M/B$ memory transfers/phase
- OPT must spend $\geq \frac{M}{2}/B$ memory transfers per phase: at best, starts phase with entire $M/2$ cache with needed items. But there are M/B blocks during phase. So $\leq$ half free

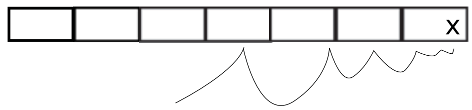
Search

Preprocess n elements in comparison model to support predecessor search for x.

B-trees

They support predecessor (and insert and delete) in $O(\log_{B+1} N)$ memory transfers.

- each node occupies $\Theta(1)$ blocks
- height$= \Theta(\log_B N)$
- need to know B

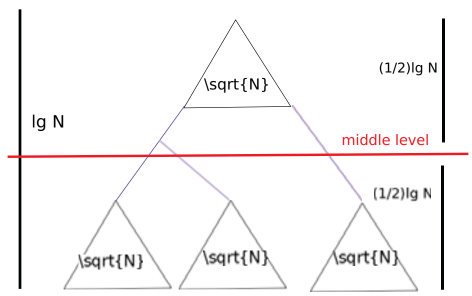


Binary search

Approximately, every iteration visits a different block until we are in x's block. Thus, $MT(N) = \Theta(\log N - \log B) = \Theta(\log(N/B))$. SLOW

van Emde Boas layout

[Prokop 1999]



- store N elements in complete BST
- carve BST at middle level of edges
- recursively layout the pieces and concatenate
- like block matrix multiplication, order of pieces doesn't matter; just need each piece to be stored consecutively

Analysis of BST search in vEB layout:

- consider recursive level of refinement at which structure has $\leq B$ nodes
- the height of the vEB tree is between $\frac{1}{2}\lg B$ and $\lg B \implies$ size is between $\sqrt{B}$ and B
  $\implies$ any root-to-node path (search path) visits $\leq \frac{\lg N}{\frac{1}{2}\lg B} = 2\log_B N$ trees that have size $\leq B$
- each tree of size $\leq B$ occupies ≤ 2 memory blocks

$\implies \leq 4\log_B N = O(\log_B N)$ memory transfers

- this generalizes to heights that are not powers of 2, B-trees of constant branching factor and dynamic B-trees: $O(\log_B N)$ insert/delete. [Bender, Demaine, Farach-Colton 2000]

Sorting

B-trees

N inserts into (cache-oblivious) B-tree $\implies MT(N) = \Theta(N \log_B N)$ NOT OPTIMAL. By contrast, BST sort is optimal $O(N \lg N)$

Binary mergesort

- binary mergesort is cache-oblivious.
- the merge is 3 parallel scans
  $\implies MT(N) = 2MT(N/2) + O(N/B + 1)$
  $MT(M) = O(M/B)$
- the recursion tree has $\lg(N/M)$ levels, and each level contributes $O(N/B)$
  $\implies MT(N) = \frac{N}{B} \lg \frac{N}{M}$. $\leftarrow \frac{B}{\lg B}$ faster than the B-tree version discussed earlier!

M/B-way mergesort

- split array into M/B equal subarrays
- recursively sort each
- merge via M/B parallel scans (keeping one "current" block per list)

$$\implies MT(N) = \frac{M}{B} MT\left(\frac{N}{M/B}\right) + O(N/B + 1)$$
$$MT(M) = O(M/B)$$

$$\begin{aligned}\implies \text{height becomes } \log_{M/B} \frac{N}{M} + 1 \\ &= \log_{M/B} \frac{N}{B} \cdot \frac{B}{M} + 1 \\ &= \log_{M/B} \frac{N}{B} - \log_{M/B} \frac{M}{B} + 1 \\ &= \log_{M/B} \frac{N}{B}\end{aligned}$$

$$\implies MT(N) = O\left(\frac{N}{B}\log_{M/B}\frac{N}{B}\right)$$

This is asymptotically optimal, in the comparison model.

Cache-oblivious Sorting

This requires the tall-cache assumption: $M = \Omega(B^{1+\epsilon})$ for some fixed $\epsilon > 0$, e.g., $M = \Omega(B^2)$ or $M/B = \Omega(B)$.

Then, $\approx N^\epsilon$-way mergesort with recursive ("funnel") merge works.

Priority Queues

- $O(\frac{1}{B}\log_{M/B}\frac{N}{B})$ per insert or delete-min
- generalizes sorting
- external memory and cache-oblivious
- see 6.851