6.046	Lecture 18	Apr. 23. 2015

TODAY: Fixed-parameter algorithms

- vertex cover
- fixed-parameter tractability
- kernelization
- connection to approximation

an alternative to approx. algorithms ~ dealing with NP-hardness

Pick any 2: (cf. friends, sleep, work)

1. hard problems
2. fast (poly.-time) algorithms
3. exact solutions

2, 3: P

1, 2: approx. algs. [L17]

1, 3: FPT [TODAY]

Idea: aim for exact algorithm, [Downey & Fellows 1997]
but isolate exponential term to a parameter
⇒ get fast solution for instances with small parameter value
- hope parameter is small in practice

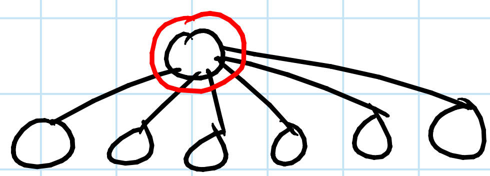
Parameter = nonnegative integer $k(x)$ ← x: problem input
- often a "natural" parameter (k in input)
- not necessarily efficiently computable (e.g. OPT)

Parameterized problem = problem + parameter
"problem w.r.t. parameter"
(potentially many interesting parameterizations)

Goal: polynomial in problem size n,
exponential in parameter k

Example: k-Vertex Cover (NP-hard)
Given: graph $G=(V,E)$, nonnegative integer k
Q: is there a set S of $\le k$ vertices
that "covers" all edges: $\forall e \in E\ \exists v \in S \cap e$
Parameter: k

Note: can have $k \ll |V|$:

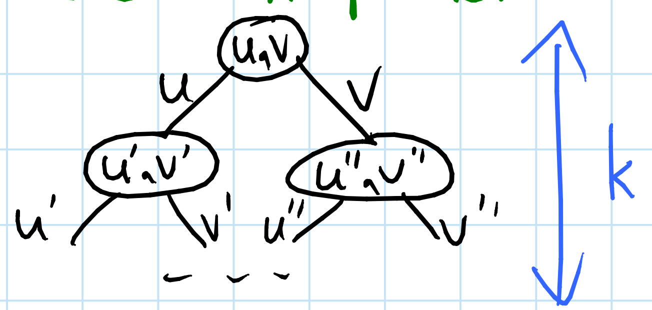


Brute-force solution: (BAD)
– try all $\binom{V}{k} + \underbrace{\binom{V}{k-1} + \cdots + \binom{V}{0}}_{\text{can skip – bigger is better}}$ sets of $\le k$ vxs.
– test coverage in $O(m)$ time (m = #edges)
$\Rightarrow O(V^k E)$ time
– polynomial for fixed k
– but not same polynomial – e.g. not $O(V^{100})$
– inefficient in most cases
$\Rightarrow$ define $n^{f(k)}$ to be BAD
↳ here $n = |V| + |E|$

general technique

Bounded search-tree algorithm: (GOOD)

- pick arbitrary edge $e=(u,v)$
- know that either $u \in S$ or $v \in S$ (or both) but don't know which
- guess: try both possibilities
  1. add u to S
     delete u & incident edges from G
     recurse with $k' = k-1$
  2. ditto with v instead of u
  - return OR of two outcomes
- like guessing in dynamic programming, but memoization doesn't help here
- recursion tree:



- at leaf ($k=0$):
  return $|E| \overset{?}{=} 0$
- $O(V)$ time to delete u or v

$\Rightarrow O(2^k \cdot V)$ time
  - $O(V)$ for fixed k
  - degree of polynomial independent of k
  - also polynomial for $k = O(\lg V)$
  - practical for e.g. $k \le 32$
  - define $f(k) \cdot n^{O(1)}$ to be GOOD

FPT: parameterized problem is **fixed-parameter tractable (FPT)** if there is an algorithm with running time $\le f(k)\, n^{O(1)}$

$f : \mathbb{N} \to \mathbb{N}$ (nonneg.) parameter indep. of k & n

Question: why $f(k) \cdot n^{O(1)}$ not $f(k) + n^{O(1)}$?

Theorem: $\exists\, f(k) \cdot n^c$ algorithm $\iff \exists\, f'(k) + n^{c'}$ algorithm

Proof: ($\Leftarrow$) trivial (assuming $f'(k)$ & $n^{c'} \ge 1$)

($\Rightarrow$) if $n \le f(k)$ then $f(k) \cdot n^c \le f(k)^{c+1}$

if $f(k) \le n$ then $f(k) \cdot n^c \le n^{c+1}$

so $f(k) \cdot n^c \le \max\{f(k)^{c+1}, n^{c+1}\}$

$\le \underbrace{f(k)^{c+1}}_{f'(k)} + n^{\underbrace{c+1}_{c'}}$. $\square$

OR: $xy \le x^2 + y^2 \to f'(k) = f(k)^2$ & $c' = 2c$

Example: $O(2^k\, n) \le O(4^k + n^2)$

Kernelization: a simplifying self-reduction

polynomial-time algorithm converting input (x,k) into small equivalent input (x',k')

$|x'| \le f(k)$ — answer(x) = answer(x')

Theorem: FPT $\Leftrightarrow$ $\exists$ kernelization

Proof: $(\Leftarrow)$ kernelize $\Rightarrow n' \le f(k)$

run any finite $g(n')$ algorithm

$\Rightarrow n^{O(1)} + g(f(k))$ time

$(\Rightarrow)$ let A be an $f(k) \cdot n^c$ algorithm

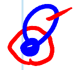
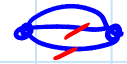
assuming k is known

- if $n \le f(k)$ then already kernelized
- if $f(k) \le n$:
  - run $A \to f(k) \cdot n^c \le n^{c+1}$ time ✓
  - output $O(1)$-size YES/NO instance as appropriate (to kernelize)

if k is unknown: run A for n^{c+1} time & if not done, know already kernelized $\square$

So (exponential) kernel exists. Recent work aims to find polynomial (even linear) kernels when possible.

Polynomial kernel for k-vertex cover:

- make graph simple:
  - remove loops  & multi-edges 
- any vertex of degree $>k$ must be in cover (else need $>k$ vertices to cover inc. edges)
- remove such vertices (& incident edges) one at a time, decreasing k accordingly

$\Rightarrow$ remaining graph has max. degree $\le k$

$\Rightarrow$ each remaining cover vertex covers $\le k$ edges

$\Rightarrow$ if #remaining edges $> k^2$, answer is No: output canonical No instance: , $\emptyset$

- else $|E'| \le k^2$
- remove isolated vertices

$\Rightarrow |V'| \le 2k^2$

$\Rightarrow$ reduced to instance (V', E') of size $O(k^2)$

quadratic kernel

- running time: $O(VE)$ obvious, $O(V+E)$ with more work
- if we now apply:
  - brute-force solution $\Rightarrow O(V+E+(2k^2)^k k^2)$ $= O(V+E+2^k k^{2k+2})$ time
  - bounded search-tree solution $\Rightarrow O(V+E+2^k k^2)$ time

Best algorithm to date: $O(kV + 1.274^k)$ [Chen, Kanj, Xia — TCS 2010]

Connection to approximation algorithms:

- take optimization problem, integral OPT
- consider associated decision problem: OPT $\le k$?
- parameterize by k

Theorem: optimization problem has $\underbrace{\text{EPTAS}}$

efficient PTAS: $f(1/\varepsilon) \cdot n^{O(1)}$

e.g. Approx-Partition [L17]

$\Rightarrow$ decision problem is FPT

Proof: (like FPTAS $\leftrightarrow$ pseudopoly. alg.)

- say maximization problem (& $\le k$ decision)
- run EPTAS with $\varepsilon = 1/2k$ in $f(2k) \cdot n^{O(1)}$
- relative error $\le 1/2k < 1/k$

$\Rightarrow$ absolute error < 1 if OPT $\le k$

- so if we find solution with value $\le k$ then OPT $\le (1 + 1/2k) \cdot k \le k + 1/2$
  integral $\Rightarrow$ OPT $\le k$ $\Rightarrow$ YES.
- else OPT $> k$ $\square$

Also: $=, \le, \ge$ decision problems are equivalent w.r.t. FPT

~ Can use this relation to prove EPTASs don't exist in some cases

6.046J / 18.410J Design and Analysis of Algorithms
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