6.055J / 2.038J The Art of Approximation in Science and Engineering
Spring 2008

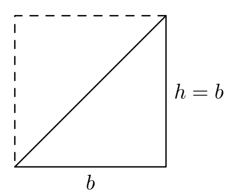
Chapter 8
Special cases

8.1 Pyramid volume

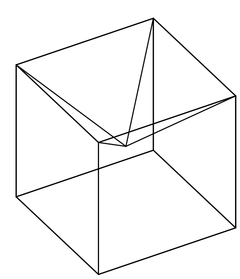
I have been promising to explain the factor of one-third in the volume of a pyramid:

$$V = \frac{1}{3}hb^2.$$

Although the method of special cases mostly cannot explain a dimensionless constant, the volume of a pyramid provides a rare counterexample.



I first explain the key idea in fewer dimensions. So, instead of immediately explaining the one-third in the volume of a pyramid, which is a difficult three-dimensional problem, first find the corresponding constant in a two-dimensional problem. That problem is the area of a triangle with base b and height h: The area is $A \sim bh$. What is the constant? Choose a convenient triangle, perhaps a 45-degree right triangle where $h = b$. Two of those triangles form a square with area b^2, so $A = b^2/2$ when $h = b$. The constant in $A \sim bh$ is therefore 1/2 *no matter what b and h are*, so $A = bh/2$.



Now use the same construction in three dimensions. What square-based pyramid, when combined with itself perhaps several times, makes a familiar shape? Only the aspect ratio h/b matters in the following discussion. So choose b conveniently, and then choose h to make a pyramid with the clever aspect ratio. The goal shape is suggested by the square pyramid base. Another solid with the same base is a cube.

Perhaps several pyramids can combine into a cube of side b. To simplify the upcoming arithmetic, I choose $b = 2$. What should the height h be? To decide, imagine how the cube will be constructed. Each cube has six faces, so six pyramids might make a cube where each pyramid base forms one face of the cube, and each pyramid tip faces inward, meeting in the center of the cube. For the tips to meet in the center of the cube, the height must be $h = 1$. So six pyramids with $b = 2$, and $h = 1$ make a cube with side length 2.

The volume of one pyramid is one-sixth of the volume of the cube:

$$V = \frac{\text{cube volume}}{6} = \frac{8}{6} = \frac{4}{3}.$$

The volume of the pyramid is $V \sim hb^2$, and the missing constant must make volume 4/3. Since $hb^2 = 4$ for these pyramids, the missing constant is 1/3. Voilà:

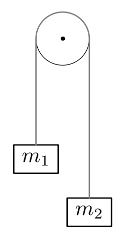
$$V = \frac{1}{3}hb^2 = \frac{4}{3}.$$

8.2 Mechanics

8.2.1 Atwood machine

The next problem illustrates dimensional analysis and special cases in a physical problem. Many of the ideas and methods from the geometry example transfer to this problem, and it introduces more methods and ways of reasoning.

The problem is a staple of first-year physics: Two masses, m_1 and m_2, are connected and, thanks to a pulley, are free to move up and down. What is the acceleration of the masses and the tension in the string? You can solve this problem with standard methods from first-year physics, which means that you can can check the solution that we derive using dimensional analysis, educated guessing, and a feel for functions.



The first problem is to find the acceleration of, say, m_1. Since m_1 and m_2 are connected by a rope, the acceleration of m_2 is, depending on your sign convention, either equal to m_1 or equal to $-m_1$. Let's call the acceleration a and use dimensional analysis to guess its form. The first step is to decide what variables are relevant. The acceleration depends on gravity, so g should be on the list. The masses affect the acceleration, so m_1 and m_2 are on the list. And that's it. You might wonder what happened to the tension: Doesn't it affect the acceleration? It does, but it is itself a consequence of m_1, m_2, and g. So adding tension to the list does not add information; it would instead make the dimensional analysis difficult.

Var	*Dim*	What
a	LT^{-2}	accel. of m_1
g	LT^{-2}	gravity
m_1	M	block mass
m_2	M	block mass

These variables fall into two pairs where the variables in each pair have the same dimensions. So there are two dimensionless groups here ripe for picking: $G_1 = m_1/m_2$ and $G_2 = a/g$. You can make any dimensionless group using these two obvious groups, as experimentation will convince you. Then, following the usual pattern,

$$\frac{a}{g} = f\left(\frac{m_1}{m_2}\right),$$

where f is a dimensionless function.

Pause a moment. The more thinking that you do to choose a clean representation, the less algebra you do later. So rather than find f using m_1/m_2 as the dimensionless group, first choose a better group. The ratio m_1/m_2 does not respect the symmetry of the problem in that only the sign of the acceleration changes when you interchange the labels m_1 and m_2. Whereas m_1/m_2 turns into its reciprocal. So the function f will have to do lots of work to turn the unsymmetric ratio m_1/m_2 into a symmetric acceleration.