6.055J / 2.038J The Art of Approximation in Science and Engineering
Spring 2008

Chapter 10
Springs

Everything is a spring! The main example in this chapter is waves, which illustrate springs, discretization, and special cases – a fitting, unified way to end the book.

10.1 Waves

Ocean covers most of the earth, and waves roam most of the ocean. Waves also cross puddles and ponds. What makes them move, and what determines their speed? By applying and extending the techniques of approximation, we analyze waves. For concreteness, this section refers mostly to water waves but the results apply to any fluid. The themes of section are: *Springs are everywhere* and *Consider limiting cases.*

10.1.1 Dispersion relations

The most organized way to study waves is to use **dispersion relations**. A dispersion relation states what values of frequency and wavelength a wave can have. Instead of the wavelength λ, dispersion relations usually connect frequency ω, and wavenumber k, which is defined as $2\pi/\lambda$. This preference has an basis in order-of-magnitude reasoning. Wavelength is the the distance the wave travels in a full period, which is 2π radians of oscillation. Although 2π is dimensionless, it is not the ideal dimensionless number, which is unity. In 1 radian of oscillation, the wave travels a distance

$$\bar{\lambda} \equiv \frac{\lambda}{2\pi}.$$

The bar notation, meaning 'divide by 2π', is chosen by analogy with h and $\hbar$. The one-radian forms such as $\hbar$ are more useful for approximations than the 2π-radian forms. The Bohr radius, in a form where the dimensionless constant is unity, contains $\hbar$ rather than h. Most results with waves are similarly simpler using $\bar{\lambda}$ rather than λ. A further refinement is to use its inverse, the wavenumber $k = 1/\bar{\lambda}$. This choice, which has dimensions of inverse length, parallels the definition of angular frequency ω, which has dimensions of inverse time. A relation that connects ω and k is likely to be simpler than one connecting ω and $\bar{\lambda}$, although either is simpler than one connecting ω and λ.

The simplest dispersion relation describes electromagnetic waves in a vacuum. Their frequency and wavenumber are related by the dispersion relation

$$\omega = ck,$$

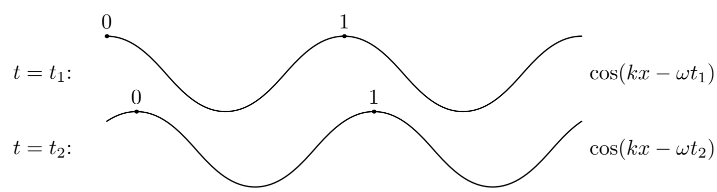
which states that waves travel at velocity $\omega/k = c$, independent of frequency. Dispersion relations contain a vast amount of information about waves. They contain, for example, how fast crests and troughs travel: the **phase velocity**. They contain how fast wave packets travel: the **group velocity**. They contain how these velocities depend on frequency: the **dispersion**. And they contain the rate of energy loss: the **attenuation**.

10.1.2 Phase and group velocities

The usual question with a wave is how fast it travels. This question has two answers, the phase velocity and the group velocity, and both depend on the dispersion relation. To get a feel for how to use dispersion relations (most of the chapter is about how to calculate them), we discuss the simplest examples that illustrate these two velocities. These analyses start with the general form of a traveling wave:

$$f(x,t) = \cos(kx - \omega t),$$

where f is its amplitude.



Phase velocity is an easier idea than group velocity so, as an example of divide-and-conquer reasoning and of maximal laziness, study it first. The phase, which is the argument of the cosine, is $kx - \omega t$. A crest occurs when the phase is zero. In the top wave, a crest occurs when $x = \omega t_1/k$. In the bottom wave, a crest occurs when $x = \omega t_2/k$. The difference

$$\frac{\omega}{k}(t_2 - t_1)$$

is the distance that the crest moved in time $t_2 - t_1$. So the phase velocity, which is the velocity of the crests, is

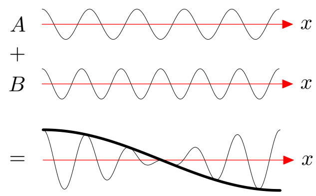
$$v_{\text{ph}} = \frac{\text{distance crest shifted}}{\text{time taken}} = \frac{\omega}{k}.$$

To check this result, check its dimensions: ω provides inverse time and $1/k$ provides length, so ω/k is a speed.

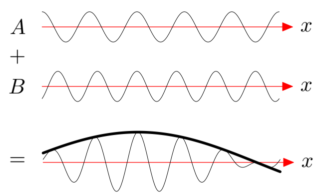
Group velocity is trickier. The word 'group' suggests that the concept involves more than one wave. Because two is the first whole number larger than one, the simplest illustration uses two waves. Instead of being a function relating ω and k, the dispersion relation here is a list of allowed (k, ω) pairs. But that form is just a discrete approximation to an official dispersion relation, complicated enough to illustrate group velocity and simple enough to not create a forest of mathematics. So here are two waves with almost the same wavenumber and frequency:

$$f_1 = \cos(kx - \omega t),$$
$$f_2 = \cos((k + \Delta k)x - (\omega + \Delta\omega)t),$$

where Δk and $\Delta\omega$ are small changes in wavenumber and frequency, respectively. Each wave has phase velocity $v_{\text{ph}} = \omega/k$, as long as Δk and $\Delta\omega$ are tiny. The figure shows their sum.



The point of the figure is that two cosines with almost the same spatial frequency add to produce an envelope (thick line). The envelope itself looks like a cosine. After waiting a while, each wave changes because of the ωt or $(\omega + \Delta\omega)t$ terms in their phases. So the sum and its envelope change to this:



The envelope moves, like the crests and troughs of any wave. It is a wave, so it has a phase velocity, which motivates the following definition:

Group velocity is the phase velocity of the envelope.

With this pictorial definition, you can compute group velocity for the wave $f_1 + f_2$. As suggested in the figures, adding two cosines produces a a slowly varying envelope times a rapidly oscillating inner function. This trigonometric identity gives the details:

$$\cos(A + B) = \underbrace{2\cos\left(\frac{B - A}{2}\right)}_{\textit{envelope}} \times \underbrace{\cos\left(\frac{A + B}{2}\right)}_{\textit{inner}}.$$

If $A \approx B$, then $A - B \approx 0$, which makes the envelope vary slowly. Apply the identity to the sum:

$$f_1 + f_2 = \cos(\underbrace{kx - \omega t}_{A}) + \cos(\underbrace{(k + \Delta k)x - (\omega + \Delta\omega)t}_{B}).$$

Then the envelope contains

$$\cos\left(\frac{B - A}{2}\right) = \cos\left(\frac{x\Delta k - t\Delta\omega}{2}\right).$$

The envelope represents a wave with phase

$$\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t.$$

So it is a wave with wavenumber $\Delta k/2$ and frequency $\Delta\omega/2$. The envelope's phase velocity is the group velocity of $f_1 + f_2$:

$$v_g = \frac{\text{frequency}}{\text{wavenumber}} = \frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k}.$$

In the limit where $\Delta k \to 0$ and $\Delta\omega \to 0$, the group velocity is

$$v_g = \frac{\partial\omega}{\partial k}.$$

It is usually different from the phase velocity. A typical dispersion relation, which appears several times in this chapter, is $\omega \propto k^n$. Then $v_{ph} = \omega/k = k^{n-1}$ and $v_g \propto nk^{n-1}$. So their ratio is

$$\frac{v_g}{v_{ph}} = n. \qquad \text{(for a power-law relation)}$$

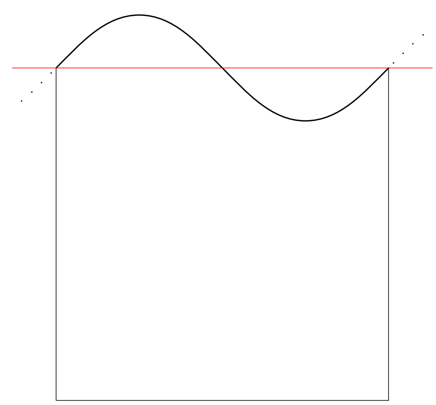
Only when $n = 1$ are the two velocities equal. Now that we can find wave velocities from dispersion relations, we return to the problem of finding the dispersion relations.

10.1.3 Dimensional analysis

A dispersion relation usually emerges from solving a wave equation, which is an unpleasant partial differential equation. For water waves, a wave equation emerges after linearizing the equations of hydrodynamics and neglecting viscosity. This procedure is mathematically involved, particularly in handling the boundary conditions. Alternatively, you can derive dispersion relations using dimensional analysis, then complete and complement the derivation with physical arguments. Such methods usually cannot evaluate the dimensionless constants, but the beauty of studying waves is that, as in most problems involving springs and oscillations, *most of these constants are unity.*

How do frequency and wavenumber connect? They have dimensions of T^{-1} and L^{-1}, respectively, and cannot form a dimensionless group without help. So include more variables. What physical properties of the system determine wave behavior? Waves on the open ocean behave differently from waves in a bathtub, perhaps because of the difference in the depth of water h. The width of the tub or ocean could matter, but then the problem becomes two-dimensional wave motion. In a first analysis, avoid that complication and consider waves that move in only one dimension, perpendicular to the width of the container. Then the width does not matter.

To determine what other variables are important, use the principle that waves are like springs, because *every physical process contains a spring*. This blanket statement cannot be strictly correct. However, it is useful as a broad generalization. To get a more precise idea of when this assumption is useful, consider the characteristics of spring motion. First, springs have an equilibrium position. If a system has an undisturbed, resting state, consider looking for a spring. For example, for waves on the ocean, the undisturbed state is a calm, flat ocean. For electromagnetic waves – springs are not confined to mechanical systems – the resting state is an empty vacuum with no radiation. Second, springs oscillate. In mechanical systems, oscillation depends on inertia to carry the mass beyond the equilibrium position. Equivalently, it depends on kinetic energy turning into potential energy, and vice versa. Water waves store potential energy in the disturbance of the surface and kinetic energy in the motion of the water. Electromagnetic waves store energy in the electric and magnetic fields. A magnetic field is generated by moving or spinning charges, so the magnetic field is a reservoir of kinetic (motion) energy. An electric field is generated by stationary charges and has an associated potential, so the electric field is the reservoir of potential energy. With these identifications, the electromagnetic field acts like a set of springs, one for each radiation frequency. These examples are positive examples. A negative example – a marble oozing its way through glycerin – illustrates that springs are not always present. The marble moves so slowly that the kinetic energy of the corn syrup, and therefore its inertia, is miniscule and irrelevant. This system has no reservoir of kinetic energy, for the kinetic energy is merely dissipated, so it does not contain a spring.



Waves have the necessary reservoirs to act like springs. The surface of water is flat in its lowest-energy state. Deviations, also known as waves, are opposed by a restoring force. Distorting the surface is like stretching a rubber sheet: Surface tension resists the distortion. Distorting the surface also requires raising the average water level, a change that gravity resists.

The average *height* of the surface does not change, but the average depth of the water does. The higher column, under the crest, has more water than the lower column, under the trough. So in averaging to find the average depth, the higher column gets a slightly higher weighting. Thus the average depth increases. This result is consistent with experience: It takes energy to make waves.

The total restoring force includes gravity and surface tension so, in the list of variables, include surface tension (γ) and gravity (g).

In a wave, like in a spring, the restoring force fights inertia, represented here by the fluid density. The gravitational piece of the restoring force does not care about density: Gravity's stronger pull on denser substances is exactly balanced by their greater inertia. This exact cancellation is a restatement of the **equivalence principle**, on which Einstein based the theory of general relativity [16, 17]. In pendulum motion, the mass of the bob drops out of the final solution for the same reason. The surface-tension piece of the restoring force, however, does not change when density changes. The surface tension itself, the fluid property γ, depends on density because it depends on the spacing of atoms at the surface. That dependence affects γ. However, once you know γ you can compute surface-tension forces without knowing the density. Since ρ does not affect the surface-tension force but affects the inertia, it affects the properties of waves in which surface tension provides a restoring force. Therefore, include ρ in the list.

Var	*Dim*	What
ω	T^{-1}	frequency
k	L^{-1}	wavenumber
g	LT^{-2}	gravity
h	L	depth
ρ	ML^{-3}	density
γ	MT^{-2}	surface tension

To simplify the analysis, assume that the waves do not lose energy. This choice excludes viscosity from the set of variables. To further simplify, exclude the speed of sound. This approximation means ignoring sound waves, and is valid as long as the flow speeds are slow compared to the speed of sound. The resulting ratio,

$$\mathcal{M} \equiv \frac{\text{flow speed}}{\text{sound speed}},$$

is dimensionless and, because of its importance, is given a name: the **Mach number**. Finally, assume that the wave amplitude ξ is small compared to its wavelength and to the depth of the container. The table shows the list of variables. Even with all these restrictions, which significantly simplify the analysis, the results explain many phenomena in the world.

These six variables built from three fundamental dimensions produce three dimensionless groups. One group is easy: the wavenumber k is an inverse length and the depth h is a length, so

$$\Pi_1 \equiv kh.$$

This group is the dimensionless depth of the water: $\Pi_1 \ll 1$ means shallow and $\Pi_1 \gg 1$ means deep water. A second dimensionless group comes from gravity. Gravity, represented by g, has the same dimensions as ω^2, except for a factor of length. Dividing by wavenumber fixes this deficit:

$$\Pi_2 = \frac{\omega^2}{gk}.$$

Without surface tension, Π_1 and Π_2 are the only dimensionless groups, and neither group contains density. This mathematical result has a physical basis. Without surface tension, the waves propagate because of gravity alone. The equivalence principle says that gravity affects motion independently of density. Therefore, density cannot – and does not – appear in either group.

Now let surface tension back into the playpen of dimensionless groups. It must belong in the third (and final) group Π_3. Even knowing that γ belongs to Π_3 still leaves great freedom in choosing its form. The usual pattern is to find the group and then interpret it, as we did for Π_1 and Π_2. Another option is to begin with a physical interpretation and use the interpretation to construct the group. Here you can construct Π_3 to measure the relative importance of surface-tension and gravitational forces. Surface tension γ has dimensions of force per length, so producing a force requires multiplying by a length. The problem already has two lengths: wavelength (represented via k) and depth. Which one should you use? The wavelength probably always affects surface-tension forces, because it determines the curvature of the surface. The depth, however, affects surface-tension forces only when it becomes comparable to or smaller than the wavelength, if even then. You can use both lengths to make γ into a force: for example, $F \sim \gamma\sqrt{h/k}$. But the analysis is easier if you use only one, in which case the wavelength is the preferable choice. So $F_\gamma \sim \gamma/k$. Gravitational force, also known as weight, is $\rho g \times$ volume. Following the precedent of using only k to produce a length, the gravitational force is $F_\text{g} \sim \rho g/k^3$. The dimensionless group is then the ratio of surface-tension to gravitational forces:

$$\Pi_3 \equiv \frac{F_\gamma}{F_\text{g}} = \frac{\gamma/k}{\rho g/k^3} = \frac{\gamma k^2}{\rho g}.$$

This choice has, by construction, a useful physical interpretation, but many other choices are possible. You can build a third group without using gravity: for example, $\Pi_3 \equiv \gamma k^3/\rho\omega^2$. With this choice, ω appears in two groups: Π_2 and Π_3. So it will be hard to solve for it. The choice made for P_3, besides being physically useful, quarantines ω in one group: a useful choice since ω is the goal.

Now use the groups to solve for frequency ω as a function of wavenumber k. You can instead solve for k as a function of ω, but the formulas for phase and group velocity are simpler with ω as a function of k. Only the group Π_2 contains ω, so the general dimensionless solution is

$$\Pi_2 = f(\Pi_1, \Pi_3),$$

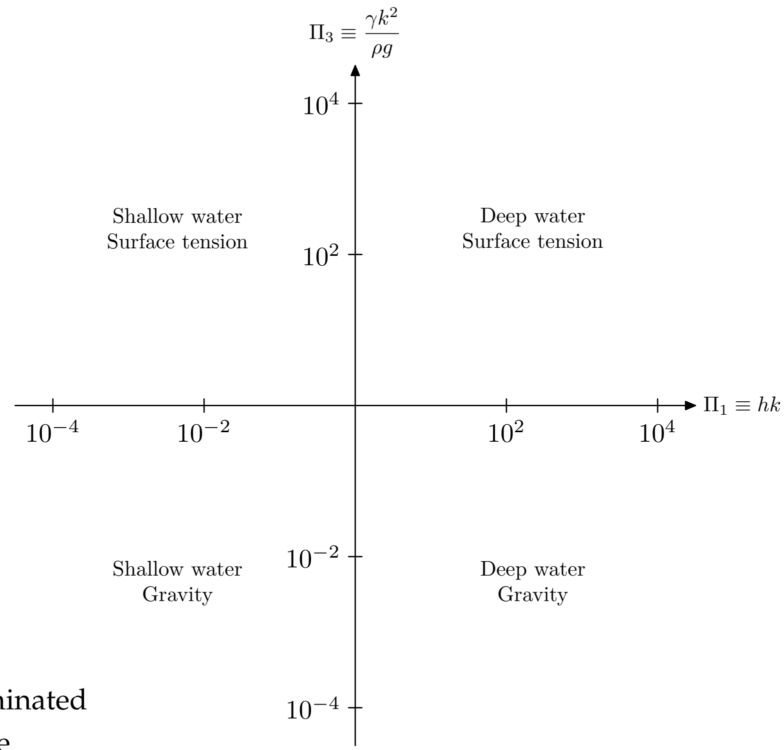
or

$$\frac{\omega^2}{gk} = f\left(kh, \frac{\gamma k^2}{\rho g}\right).$$

Then

$$\omega^2 = gk \cdot f(kh, \frac{\gamma k^2}{\rho g}).$$

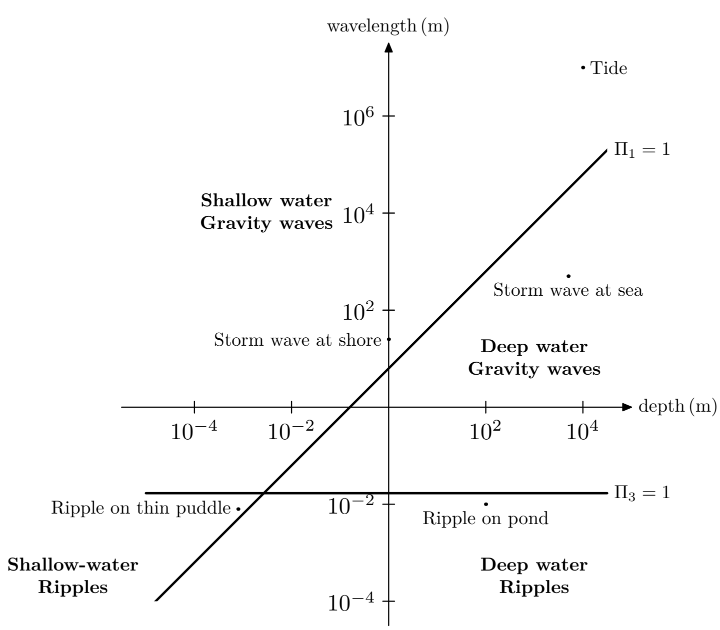
This relation is valid for waves in shallow or deep water (small or large Π_1); for waves propagated by gravity or by surface tension (small or large Π_3); and for waves in between.



The figure shows how the two groups Π_1 and Π_3 divide the world of waves into four regions. We study each region in turn, and combine the analyses to understand the whole world (of waves). The group Π_1 measures the depth of the water: Are the waves moving on a puddle or an ocean? The group Π_3 measures the relative contribution of surface tension and gravity: Are the waves ripples or gravity waves?

The division into deep and shallow water (left and right sides) follows from the interpretation of $\Pi_1 = kh$ as dimensionless depth. The division into surface-tension- and gravity-dominated waves (top and bottom halves) is more subtle, but is a result of how Π_3 was constructed. As a check, look at Π_3. Large g or small k result in the same consequence: small Π_3. Therefore the physical consequence of longer wavelength (smaller k) is similar to that of stronger gravity. So longer-wavelength waves are gravity waves. The large-Π_3 portion of the world (top half) is therefore labeled with surface tension.

The next figure shows how wavelength and depth (instead of the dimensionless groups) partition the world, and plots examples of different types of waves.



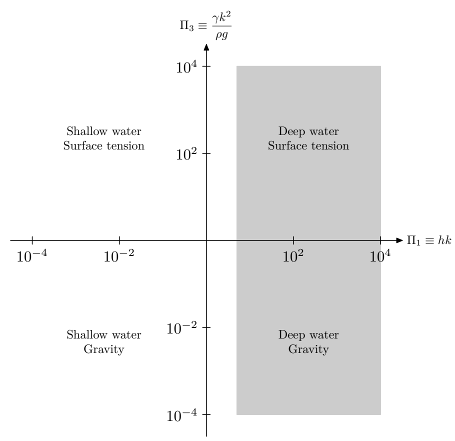
The thick dividing lines are based on the dimensionless groups $\Pi_1 = hk$ and $\Pi_3 = \gamma k^2/\rho g$. Each region contains one or two examples of its kind of waves. With $g = 1000\,\text{cm}\,\text{s}^{-1}$

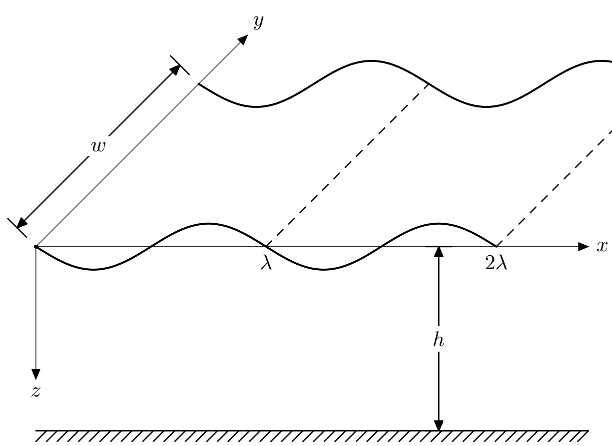
and $\rho \sim 1\,\mathrm{g\,cm^{-3}}$, the border wavelength between ripples and gravity waves is just over $\lambda \sim 1\,\mathrm{cm}$ (the horizontal, $\Pi_3 = 1$ dividing line).

The magic function f is still unknown to us. To determine its form and to understand its consequences, study f in limiting cases. Like a jigsaw-puzzle-solver, study first the corners of the world, where the physics is simplest. Then connect the corner solutions to get solutions valid along an edge, where the physics is the almost as simple as in a corner. Finally, crawl inward to assemble the complicated, complete solution. This extended example illustrates divide-and-conquer reasoning, and using limiting cases to choose pieces into which you break the problem.

10.1.4 Deep water

First study deep water, where $kh \gg 1$, as shaded in the map. Deep water is defined as water sufficiently deep that waves cannot feel the bottom of the ocean. How deep do waves' feelers extend? The only length scale in the waves is the wavelength, $\lambda = 2\pi/k$. The feelers therefore extend to a depth $d \sim 1/k$ (as always, neglect constants, such as 2π). This educated guess has a justification in Laplace's equation, which is the spatial part of the wave equation. Suppose that the waves are periodic in the x direction, and z measures depth below the surface, as shown in this figure:





Then, Laplace's equation becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0,$$

where ϕ is the velocity potential. The $\partial^2\phi/\partial y^2$ term vanishes because nothing varies along the width (the y direction).

It's not important what exactly ϕ is. You can find out more about it in an excellent fluid-mechanics textbook, *Fluid Dynamics for Physicists* [18]; Lamb's *Hydrodynamics* [19] is a classic but difficult. For this argument, all that matters is that ϕ measures the effect of the wave and that ϕ satisfies Laplace's equation. The wave is periodic in the x direction, with a form such as $\sin kx$. Take

$$\phi \sim Z(z) \sin kx.$$

The function $Z(z)$ measures how the wave decays with depth.

The second derivative in x brings out two factors of k, *and a minus sign*:

$$\frac{\partial^2 \phi}{\partial x^2} = -k^2 \phi.$$

In order that this ϕ satisfy Laplace's equation, the z-derivative term must contribute $+k^2\phi$. Therefore,

$$\frac{\partial^2 \phi}{\partial z^2} = k^2 \phi,$$

so $Z(z) \sim e^{\pm kz}$. The physically possible solution – the one that does not blow up exponentially at the bottom of the ocean – is $Z(z) \sim e^{-kz}$. Therefore, relative to the effect of the wave at the surface, the effect of the wave at the bottom of the ocean is $\sim e^{-kh}$. When $kh \gg 1$, the bottom might as well be on the moon because it has no effect. The dimensionless factor kh – it must be dimensionless to sit alone in an exponent – compares water depth with feeler depth $d \sim 1/k$:

$$\frac{\text{water depth}}{\text{feeler depth}} \sim \frac{h}{1/k} = hk,$$

which is the dimensionless group Π_1.

In deep water, where the bottom is hidden from the waves, the water depth h does not affect their propagation, so h disappears from the list of relevant variables. When h goes, so does $\Pi_1 = kh$. There is one caveat. If Π_1 is the only group that contains k, then you cannot blithely discard Π_1 just because you no longer care about h. If you did, you would be discarding k and h, and make it impossible to find a dispersion relation (which connects ω and k). Fortunately, k appears in $\Pi_3 = \gamma k^2/\rho g$ as well as in Π_1. So in deep water it is safe to discard Π_1. This argument for the irrelevance of h is a physical argument. It has a mathematical equivalent in the language of dimensionless groups and functions. Because h has dimensions, the statement that 'h is large' is meaningless. The question is, 'large compared to what length?' With $1/k$ as the standard of comparison the meaningless 'h is large' statement becomes 'kh is large.' The product kh is the dimensionless group Π_1. Mathematically, you are assuming that the function $f(kh, \gamma k^2/\rho g)$ has a limit as $kh \to \infty$.

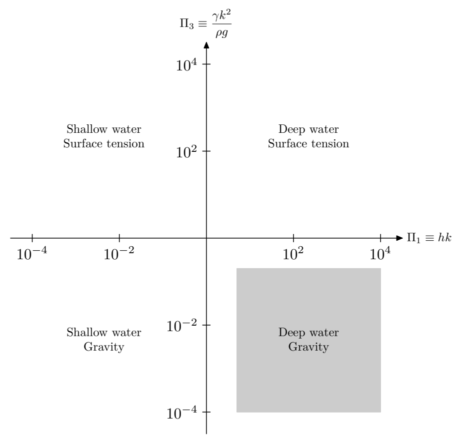
Without Π_1, the general dispersion relation simplifies to

$$\omega^2 = gk\, f_{\text{deep}}\left(\frac{\gamma k^2}{\rho g}\right).$$

This relation describes the deep-water edge of the world of waves. The edge has two corners, labeled by whether gravity or surface tension provides the restoring force. Although the form of f_{deep} is unknown, it is a simpler function than the original f, a function of two variables. To determine the form of f_{deep}, continue the process of dividing and conquering: Partition deep-water waves into its two limiting cases, gravity waves and ripples.

10.1.5 Gravity waves on deep water

Of the two extremes, gravity waves are the more common. They include wakes generated by ships and most waves generated by wind. So specialize to the corner of the wave world where water is deep and gravity is strong. With gravity much stronger than surface tension, the dimensionless group $\Pi_3 = \gamma k^2/\rho g$ limits to 0. Since Π_3 is the product of several factors, you can achieve the limit in several ways:



1. Increase g (which is downstairs) by moving to Jupiter.
2. Reduce γ (which is upstairs) by dumping soap on the water.
3. Reduce k (which is upstairs) by studying waves with a huge wavelength.

In this limit, the general deep-water dispersion relation simplifies to

$$\omega^2 = f_{\text{deep}}(0)gk = C_1 gk,$$

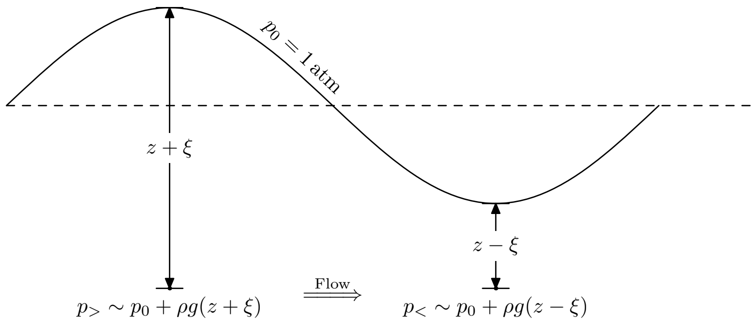
where $f_{\text{deep}}(0)$ is an as-yet-unknown constant, C_1. The use of $f_{\text{deep}}(0)$ assumes that $f_{\text{deep}}(x)$ has a limit as $x \to 0$. The slab argument, which follows shortly, shows that it does. For now, in order to make progress, assume that it has a limit. The constant remains unknown to the lazy methods of dimensional analysis, because the methods sacrifice evaluation of dimensionless constants to gain comprehension of physics. Usually assume that such constants are unity. In this case, we get lucky: An honest calculation produces $C_1 = 1$ and

$$\omega^2 = 1\times gk,$$

where the red $1\times$ indicates that it is obtained from honest physics.

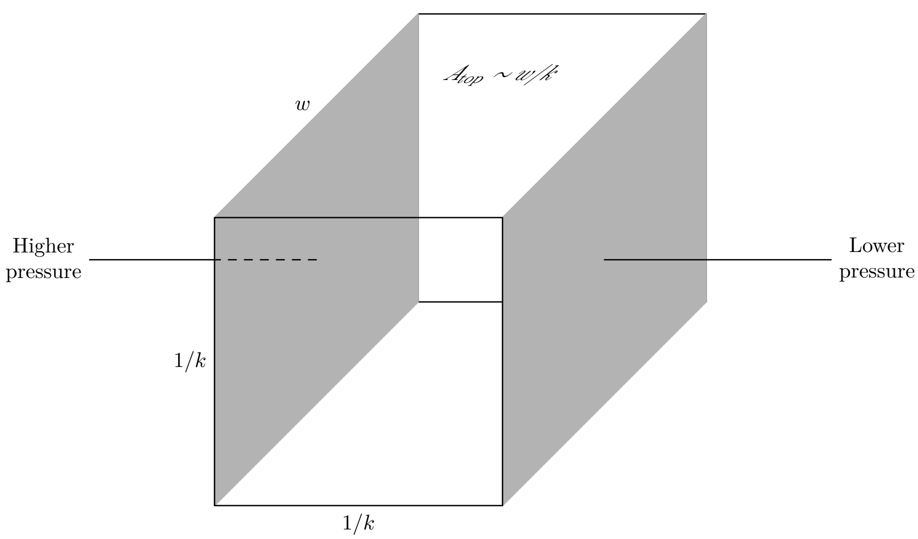
Such results from dimensional analysis seem like rabbits jumping from a hat. The dispersion relation is correct, but your gut may grumble about this magical derivation and ask, ‘But *why* is the result true?’ A physical model of the forces or energies that drive the waves explains the origin of the dispersion relation. The first step is to understand the mechanism: How does gravity make the water level rise and fall? Taking a hint from the Watergate investigators,[1] we follow the water. The water in the crest does *not* move into the trough. Rather, the water in the crest, being higher, creates a pressure underneath it higher than that of the water in the trough, as shown in this figure:

[1] When the reporters Woodward and Bernstein [20] were investigating criminal coverups during the Nixon administration, they received help from the mysterious ‘Deep Throat’, whose valuable advice was to ‘follow the money.’



The higher pressure forces water underneath the crest to flow toward the trough, making the water level there rise. Like a swing sliding past equilibrium, the surface overshoots the equilibrium level to produce a new crest and the cycle repeats.

The next step is to quantify the model by estimating sizes, forces, speeds, and energies. In **Section 9.1** we analyzed a messy mortality curve by replacing it with a more tractable shape: a rectangle. The method of discretization worked there, so try it again. 'A method is a trick I use twice.'
—George Polyà. Water just underneath the surface moves quickly because of the pressure gradient. Farther down, it moves more slowly. Deep down it does not move at all. Replace this smooth falloff with a step function: Pretend that water down to a certain depth moves as a block, while deeper water stays still:



How deep should this slab of water extend? By the Laplace-equation argument, the pressure variation falls off exponentially with depth, with length scale $1/k$. So assume that the slab has a similar length scale, that it has depth $1/k$. What choice do you have? On an infinitely deep ocean, the only length scale is $1/k$. How long should the slab be? Its length should be roughly the peak-to-trough distance of the wave because the surface height changes significantly over that distance. This distance is $1/k$. Actually, it is π/k (one-half of a period), but ignore constants. All the constants combine into a giant constant at the end, which dimensional analysis cannot determine anyway, so discard it now! The slab's width w is arbitrary and cancels by the end of any analysis.

So the slab of water has depth $1/k$, length $1/k$, and width w. Estimate the forces acting on it by estimating the pressure gradients. Across the width of the slab (the y direction), the water surface is level, so the pressure is constant along the width. Into the depths (the z direction), the pressure varies because of gravity – the $\rho g h$ term from hydrostatics – but that variation is just sufficient to prevent the slab from sinking. We care about only the pressure difference across the length, the direction that the wave moves. This pressure difference depends on the height of the crest, ξ and is $\Delta p \sim \rho g \xi$. This pressure difference acts on a cross-section with area $A \sim w/k$ to produce a force

$$F \sim \underbrace{w/k}_{\text{area}} \times \underbrace{\rho g \xi}_{\Delta p} = \rho g w \xi / k.$$

The slab has mass

$$m = \rho \times \underbrace{w/k^2}_{\text{volume}},$$

so the force produces an acceleration

$$a_{\text{slab}} \sim \underbrace{\frac{\rho g w \xi}{k}}_{\text{force}} \bigg/ \underbrace{\frac{\rho w}{k^2}}_{\text{mass}} = g \xi k.$$

The factor of g says that the gravity produces the acceleration. Full gravitational acceleration is reduced by the dimensionless factor ξk, which is roughly the slope of the waves.

The acceleration of the slab determines the acceleration of the surface. If the slab moves a distance x, it sweeps out a volume of water $V \sim xA$. This water moves under the trough, and forces the surface upward a distance V/A_{top}. Because $A_{\text{top}} \sim A$ (both are $\sim w/k$), the surface moves the same distance x that the slab moves. Therefore, the slab's acceleration a_{slab} equals the acceleration a of the surface:

$$a \sim a_{\text{slab}} \sim g \xi k.$$

This equivalence of slab and surface acceleration does not hold in shallow water, where the bottom at depth h cuts off the slab before $1/k$; that story is told in **Section 10.1.12**.

The slab argument is supposed to justify the deep-water dispersion relation derived by dimensional analysis. That relation contains frequency whereas acceleration relation does not. So massage it until ω appears. The acceleration relation contains a and ξ, whereas the dispersion relation does not. An alternative expression for the acceleration might make the acceleration relation more like the dispersion relation. With luck the expression will contain ω^2, thereby producing the hoped-for ω^2; as a bonus, it will contain ξ to cancel the ξ in the acceleration relation.

In simple harmonic motion (springs!), acceleration is $a \sim \omega^2 \xi$, where ξ is the amplitude. In waves, which behave like springs, a is given by the same expression. Here's why. In time $\tau \sim 1/\omega$, the surface moves a distance $d \sim \xi$, so $a/\omega^2 \sim \xi$ and $a \sim \omega^2\xi$. With this replacement, the acceleration relation becomes

$$\underbrace{\omega^2\xi}_{a} \sim g\xi k,$$

or

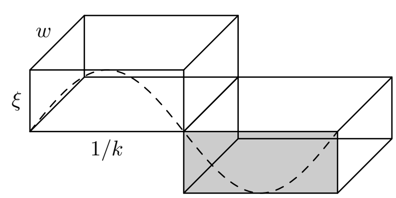
$$\omega^2 = 1\times gk,$$

which is the longed-for dispersion relation with the correct dimensionless constant in red.

An exact calculation confirms the usual hope that the missing dimensionless constants are close to unity, or are unity. This fortune suggests that the procedures for choosing how to measure the lengths were reasonable. The derivation depended on two choices:

1. Replacing an exponentially falling variation in velocity potential by a step function with size equal to the length scale of the exponential decay.
2. Taking the length of the slab to be $1/k$ instead of π/k. This choice uses only 1 radian of the cycle as the characteristic length, instead of using a half cycle or π radians. Since 1 is a more natural dimensionless number than π is, choosing 1 radian rather than π or 2π radians often improves approximations.

Both approximations are usually accurate in order-of-magnitude calculations. Rarely, however, you will get caught by a factor of $(2\pi)^6$, and wish that you had used a full cycle instead of only 1 radian.



The derivation that resulted in the dispersion relation analyzed the motion of the slab using forces. Another derivation of it uses energy by balancing kinetic and potential energy. To make a wavy surface requires energy, as shown in the figure. The crest rises a characteristic height ξ above the zero of potential, which is the level surface. The volume of water moved upward is $\xi w/k$. So the potential energy is

$$\text{PE}_{\text{gravity}} \sim \underbrace{\rho\xi w/k}_{m}\times g\xi \sim \rho g w\xi^2/k.$$

The kinetic energy is contained in the sideways motion of the slab and in the upward motion of the water pushed by the slab. The slab and surface move at the same speed; they also have the same acceleration. So the sideways and upward motions contribute similar energies. If you ignore constants such as 2, you do not need to compute the energy contributed by both motions and can do the simpler computation, which is the sideways motion. The surface moves a distance ξ in a time $1/\omega$, so its velocity is $\omega\xi$. The slab has the same speed (except for constants) as the surface, so the slab's kinetic energy is

$$\mathrm{KE_{deep}} \sim \underbrace{\rho w/k^2}_{m_\mathrm{slab}} \times \underbrace{\omega^2\xi^2}_{v^2} \sim \rho\omega^2\xi^2 w/k^2.$$

This energy balances the potential energy

$$\underbrace{\rho\omega^2\xi^2 w/k^2}_{KE} \sim \underbrace{\rho g w\xi^2/k}_{PE}.$$

Canceling the factor $\rho w\xi^2$ (in red) common to both energies leaves

$$\omega^2 \sim gk.$$

The energy method agrees with the force method, as it should, because energy can be derived from force by integration. The energy derivation gives an interpretation of the dimensionless group Π_2:

$$\Pi_2 \sim \frac{\text{kinetic energy in slab}}{\text{gravitational potential energy}} \sim \frac{\omega^2}{gk}.$$

The gravity-wave dispersion relation $\omega^2 = gk$ is equivalent to $\Pi_2 \sim 1$, or to the assertion that kinetic and gravitational potential energy are comparable in wave motion. This rough equality is no surprise because waves are like springs. In spring motion, kinetic and potential energies have equal averages, a consequence of the virial theorem.

The dispersion relation was derived in three ways: by dimensional analysis, energy, and force. Using multiple methods increases our confidence not only in the result but also in the methods. 'I have said it thrice: What I tell you three times is true.'
–Lewis Carroll, *Hunting of the Snark*.

We gain confidence in the methods of dimensional analysis and in the slab model for waves. If we study nonlinear waves, for example, where the wave height is no longer infinitesimal, we can use the same techniques along with the slab model with more confidence.

With reasonable confidence in the dispersion relation, it's time study its consequences: the phase and group velocities. The crests move at the phase velocity: $v_\mathrm{ph} = w/k$. For deep-water gravity waves, this velocity becomes

$$v_\mathrm{ph} = \sqrt{\frac{g}{k}},$$

or, using the dispersion relation to replace k by ω,

$$v_\mathrm{ph} = \frac{g}{\omega}.$$

Let's check upstairs and downstairs. Who knows where ω belongs, but g drives the waves so it should and does live upstairs.

In an infinite, single-frequency wave train, the crests and troughs move at the phase speed. However, a finite wave train contains a mixture of frequencies, and the various frequencies move at different speeds as given by

$$v_{\mathrm{ph}} = \frac{g}{\omega}.$$

Deep water is **dispersive**. Dispersion makes a finite wave train travel with the group velocity, given by $v_{\mathrm{g}} = \partial w/\partial k$, as explained in **Section 10.1.2**. The group velocity is

$$v_{\mathrm{g}} = \frac{\partial}{\partial k}\sqrt{gk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}v_{\mathrm{ph}}.$$

So the group velocity is one-half of the phase velocity, as the result for power-law dispersion relation predicts. Within a wave train, the crests move at the phase velocity, twice the group velocity, shrinking and growing to fit under the slower-moving envelope.

An everyday consequence is that ship wakes trail the ship. A ship moving with velocity v creates gravity waves with $v_{\mathrm{ph}} = v$. The waves combine to produce wave trains that propagate forward with the group velocity, which is only $v_{\mathrm{ph}}/2 = v/2$. From the ship's point of view, these gravity waves travel backward. In fact, they form a wedge, and the opening angle of the wedge depends on the one-half that arises from the exponent.

10.1.6 Surfing

Let's apply the dispersion relation to surfing. Following one winter storm reported in the *Los Angeles Times* – the kind of storm that brings cries of 'Surf's up!' – waves arrived at Los Angeles beaches roughly every 18 s. How fast were the storm winds that generated the waves? Wind pushes the crests as long as they move more slowly than the wind. After a long-enough push, the crests move with nearly the wind speed. Therefore the phase velocity of the waves is an accurate approximation to the wind speed.

The phase velocity is g/ω. In terms of the wave period T, this velocity is $v_{\mathrm{ph}} = gT/2\pi$, so

$$v_{\mathrm{wind}} \sim v_{\mathrm{ph}} \sim \frac{\overbrace{10\,\mathrm{m\,s^{-2}}}^{g} \times \overbrace{18\,\mathrm{s}}^{T}}{2\times 3} \sim 30\,\mathrm{m\,s^{-1}}.$$

In units more familiar to Americans, this wind speed is 60 mph, which is a strong storm: about 10 on the Beaufort wind scale ('whole gale/storm'). The wavelength is given by

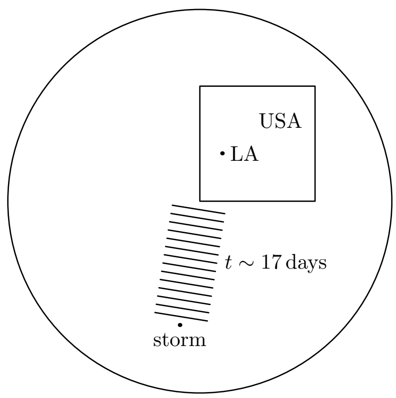
$$\lambda = v_{\mathrm{ph}}T \sim 30\,\mathrm{m\,s^{-1}} \times 18\,\mathrm{s} \sim 500\,\mathrm{m}.$$

On the open ocean, the crests are separated by half a kilometer. Near shore they bunch up because they feel the bottom; this bunching is a consequence of the shallow-water dispersion relation, the topic of **Section 10.1.13**.

In this same storm, the waves arrived at 17 s intervals the following day: a small decrease in the period. Before racing for the equations, first check that this decrease in period is reasonable. This precaution is a sanity check. If the theory is wrong about a physical effect as fundamental as a sign – whether the period should decrease or increase – then it neglects important physics. The storm winds generate waves of different wavelengths and periods, and the different wavelengths sort themselves during the trip from the far ocean to Los Angeles. Group and phase velocity are proportional to $1/\omega$, which is proportional to the period. So longer-period waves move faster, and the 18 s waves should arrive before the 17 s waves. They did! The decline in the interval allows us to calculate the distance to the storm. In their long journey, the 18 s waves raced 1 day ahead of the 17 s waves. The ratio of their group velocities is

$$\frac{\text{velocity(18 s waves)}}{\text{velocity(17 s waves)}} = \frac{18}{17} = 1 + \frac{1}{17}.$$

so the race must have lasted roughly $t \sim 17\,\text{days} \sim 1.5 \cdot 10^6\,\text{s}$. The wave train moves at the group velocity, $v_g = v_{ph}/2 \sim 15\,\text{m}\,\text{s}^{-1}$, so the storm distance was $d \sim tv_g \sim 2 \cdot 10^4\,\text{km}$, or roughly halfway around the world, an amazingly long and dissipation-free journey.



10.1.7 Speedboating

Our next application of the dispersion relation is to speedboating: How fast can a boat travel? We exclude hydroplaning boats from our analysis (even though some speedboats can hydroplane). Longer boats generally move faster than shorter boats, so it is likely that the length of the boat, l, determines the top speed. The density of water might matter. However, v (the speed), ρ, and l cannot form a dimensionless group. So look for another variable. Viscosity is irrelevant because the Reynolds number for boat travel is gigantic. Even for a small boat of length 5 m, creeping along at $2\,\text{m}\,\text{s}^{-1}$,

$$Re \sim \frac{500\,\text{cm} \times 200\,\text{cm}\,\text{s}^{-1}}{10^{-2}\,\text{cm}^2\,\text{s}^{-1}} \sim 10^7.$$

At such a huge Reynolds number, the flow is turbulent and nearly independent of viscosity (**Section 8.3.7**). Surface tension is also irrelevant, because boats are much longer than a ripple wavelength (roughly 1 cm). The search for new variables is not meeting with success. Perhaps gravity is relevant. The four variables v, ρ, g, and l, build from three dimensions, produce one dimensionless group: v^2/gl, also called the **Froude number**:

$$\text{Fr} \equiv \frac{v^2}{gl}.$$

The critical Froude number, which determines the maximum boat speed, is a dimensionless constant. As usual, we assume that the constant is unity. Then the maximum boating speed is:

$$v \sim \sqrt{gl}.$$

A rabbit has jumped out of our hat. What physical mechanism justifies this dimensional-analysis result? Follow the waves as a boat plows through water. The moving boat generates waves (the wake), and it rides on one of those waves. Take the bow wave: It is a gravity wave with $v_{\text{ph}} \sim v_{\text{boat}}$. Because $v_{\text{ph}}^2 = \omega^2/k^2$, the dispersion relation tells us that

$$v_{\text{boat}}^2 \sim \frac{\omega^2}{k^2} = \frac{g}{k} = g\bar{\lambda},$$

where $\bar{\lambda} \equiv 1/k = \lambda/2\pi$. So the wavelength of the waves is roughly v_{boat}^2/g. The other length in this problem is the boat length; so the Froude number has this interpretation:

$$\text{Fr} = \frac{v_{\text{boat}}^2/g}{l} \sim \frac{\text{wavelength of bow wave}}{\text{length of boat}}.$$

Why is $\text{Fr} \sim 1$ the critical number, the assumption in finding the maximum boat speed? Interesting and often difficult physics occurs when a dimensionless number is near unity. In this case, the physics is as follows. The wave height changes significantly in a distance $\bar{\lambda}$; if the boat's length l is comparable to $\bar{\lambda}$, then the boat rides on its own wave and tilts upward. Tilting upward, it presents a large cross-section to the water, and the drag becomes huge. [Catamarans and hydrofoils skim the water, so this kind of drag does not limit their speed. The hydrofoil makes a much quicker trip across the English channel than the ferry makes, even though the hydrofoil is much shorter.] So the top speed is given by

$$v_{\text{boat}} \sim \sqrt{gl}.$$

For a small motorboat, with length $l \sim 5\,\text{m}$, this speed is roughly $7\,\text{m}\,\text{s}^{-1}$, or 15 mph. Boats (for example police boats) do go faster than the nominal top speed, but it takes plenty of power to fight the drag, which is why police boats have huge engines.

The Froude number in surprising places. It determines, for example, the speed at which an animal's gait changes from a walk to a trot or, for animals that do not trot, to a run. In **Section 10.1.7** it determines maximum boating speed. The Froude number is a ratio of potential energy to kinetic energy, as massaging the Froude number shows:

$$\text{Fr} = \frac{v^2}{gl} = \frac{mv^2}{mgl} \sim \frac{\text{kinetic energy}}{\text{potential energy}}.$$

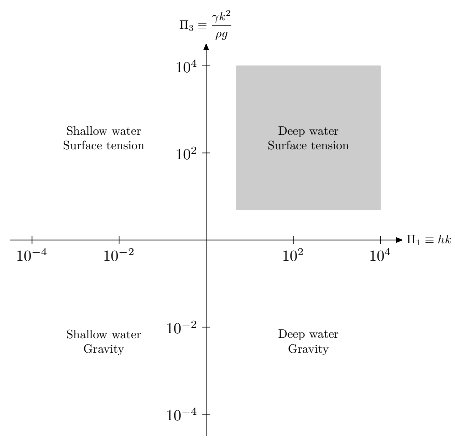
Here the massage technique was multiplication by unity (in red). In this example, the length l is a horizontal length, so gl is not a gravitational energy, but it has a similar structure and in other examples often has an easy interpretation as gravitational energy.

10.1.8 Walking

In the Froude number for walking speed, l is leg length, and gl is a potential energy. For a human with leg length $l \sim 1\,\mathrm{m}$, the condition $\mathrm{Fr} \sim 1$ implies that $v \sim 3\,\mathrm{m\,s^{-1}}$ or 6 mph. This speed is a rough estimate for the top speed for a race walker. The world record for men's race walking was once held by Bernado Segura of Mexico. He walked 20 km in 1h:17m:25.6s, for a speed of $4.31\,\mathrm{m\,s^{-1}}$.

This example concludes the study of gravity waves on deep water, which is one corner of the world of waves.

10.1.9 Ripples on deep water



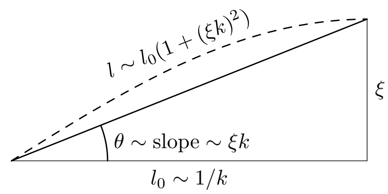
For small wavelengths (large k), surface tension rather than gravity provides the restoring force. This choice brings us to the shaded corner of the figure. If surface tension rather than gravity provides the restoring force, then g vanishes from the final dispersion relation. How to get rid of g and find the new dispersion relation? You could follow the same pattern as for gravity waves (**Section 10.1.5**). In that situation, the surface tension γ was irrelevant, so we discarded the group $\Pi_3 \equiv \gamma k^2/\rho g$. Here, with g irrelevant you might try the same trick: Π_3 contains g so discard it. In that argument lies infanticide, because it also throws out the physical effect that determines the restoring force, namely surface tension. To retrieve the baby from the bathwater, you cannot throw out $\gamma k^2/\rho g$ directly. Instead you have to choose the form of the dimensionless function f_{deep} in so that only gravity vanishes from the dispersion relation.

The deep-water dispersion relation contains one power of g in front. The argument of f_{deep} also contains one power of g, in the denominator. If f_{deep} has the form $f_{\mathrm{deep}}(x) \sim x$, then g cancels. With this choice, the dispersion relation is

$$\omega^2 = 1\times\frac{\gamma k^3}{\rho}.$$

Again the dimensionless constant from exact calculation (in red) is unity, which we would have assumed anyway. Let's reuse the slab argument to derive this relation.

In the slab picture, replace gravitational by surface-tension energy, and again balance potential and kinetic energies. The surface of the water is like a rubber sheet. A wave disturbs the surface and stretches the sheet. This stretching creates area ΔA and therefore requires energy $\gamma\Delta A$. So to estimate the energy, estimate the extra area that a wave of amplitude ξ and wavenumber k creates. The extra area depends on the extra length in a sine wave compared to a flat line. The typical slope in the sine wave $\xi \sin kx$ is ξk. Instead of integrating to find the arc length, you can approximate the curve as a straight line with slope ξk:



Relative to the level line, the tilted line is longer by a factor $1 + (\xi k)^2$.

As before, imagine a piece of a wave, with characteristic length $1/k$ in the x direction and width w in the y direction. The extra area is

$$\Delta A \sim \underbrace{w/k}_{\textit{level area}} \times \underbrace{(\xi k)^2}_{\textit{fractional increase}} \sim w\xi^2 k.$$

The potential energy stored in this extra surface is

$$\text{PE}_{\text{ripple}} \sim \gamma \Delta A \sim \gamma w \xi^2 k.$$

The kinetic energy in the slab is the same as it is for gravity waves, which is:

$$\text{KE} \sim \rho\omega^2\xi^2 w/k^2.$$

Balancing the energies

$$\underbrace{\rho\omega^2\xi^2 w/k^2}_{\textit{KE}} \sim \underbrace{\gamma w \xi^2 k}_{\textit{PE}},$$

gives

$$\omega^2 \sim \gamma k^3/\rho.$$

This dispersion relation agrees with the result from dimensional analysis. For deep-water gravity waves, we used both energy and force arguments to re-derive the dispersion relation. For ripples, we worked out the energy argument, and you are invited to work out the corresponding force argument.

The energy calculation completes the interpretations of the three dimensionless groups. Two are already done: Π_1 is the dimensionless depth and Π_2 is ratio of kinetic energy to gravitational potential energy. We constructed Π_3 as a group that compares the effects of surface tension and gravity. Using the potential energy for gravity waves and for ripples, the comparison becomes more precise:

$$\begin{aligned}\Pi_3 &\sim \frac{\text{potential energy in a ripple}}{\text{potential energy in a gravity wave}}\\ &\sim \frac{\gamma w \xi^2 k}{\rho g w \xi^2/k}\\ &\sim \frac{\gamma k^2}{\rho g}.\end{aligned}$$

Alternatively, Π_3 compares $\gamma k^2/\rho$ with g:

$$\Pi_3 \equiv \frac{\gamma k^2/\rho}{g}.$$

This form of Π_3 may seem like a trivial revision of $\gamma k^2/\rho g$. However, it suggests an interpretation of surface tension: that surface tension acts like an effective gravitational field with strength

$$g_{\text{surface tension}} = \gamma k^2/\rho,$$

In a balloon, the surface tension of the rubber implies a higher pressure inside than outside. Similarly in wave the water skin implies a higher pressure underneath the crest, which is curved like a balloon; and a lower pressure under the trough, which is curved opposite to a balloon. This pressure difference is just what a gravitational field with strength $g_{\text{surface tension}}$ would produce. This trick of effective gravity, which we used for the buoyant force on a falling marble (**Section 8.3.4**), is now promoted to a method (a trick used twice).

So replace g in the gravity-wave potential energy with this effective g to get the ripple potential energy:

$$\underbrace{\rho g w \xi^2/k}_{\textit{PE(gravity wave)}} \xrightarrow{g\to\gamma k^2/\rho} \underbrace{\gamma w \xi^2 k}_{\textit{PE(ripple)}}.$$

The left side becomes the right side after making the substitution above the arrow. The same replacement in the gravity-wave dispersion relation produces the ripple dispersion relation:

$$\omega^2 = gk \quad \xrightarrow{g\to\gamma k^2/\rho} \quad \omega^2 = \frac{\gamma k^3}{\rho}.$$

The interpretation of surface tension as effective gravity is useful when we combine our solutions for gravity waves and for ripples, in **Section 10.1.11** and **Section 10.1.16**. Surface tension and gravity are symmetric: We could have reversed the analysis and interpreted gravity as effective surface tension. However, gravity is the more familiar force, so we use effective gravity rather than effective surface tension.

With the dispersion relation you can harvest the phase and group velocities. The phase velocity is

$$v_{\text{ph}} \equiv \frac{\omega}{k} = \sqrt{\frac{\gamma k}{\rho}},$$

and the group velocity is

$$v_{\text{g}} \equiv \frac{\partial\omega}{\partial k} = \frac{3}{2}v_{\text{ph}}.$$

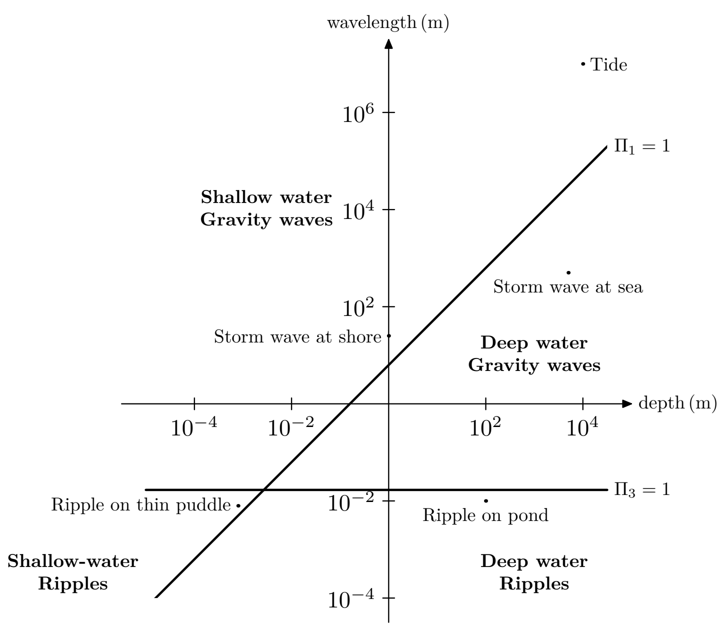
The factor of 3/2 is a consequence of the form of the dispersion relation: $\omega \propto k^{3/2}$; for gravity waves, $\omega \propto k^{1/2}$, and the corresponding factor is 1/2. In contrast to deep-water waves, a train of ripples moves *faster* than the phase velocity. So, ripples steam ahead of a boat, whereas gravity waves trail behind.

10.1.10 Typical ripples

Let's work out speeds for typical ripples, such as the ripples from dropping a pebble into a pond. From observation, these ripples have wavelength $\lambda \sim 1\,\mathrm{cm}$, and therefore wavenumber $k = 2\pi/\lambda \sim 6\,\mathrm{cm}^{-1}$. The surface tension of water (??) is $\gamma \sim 0.07\,\mathrm{J\,m^{-2}}$. So the phase velocity is

$$v_{\text{ph}} = \left(\frac{\overbrace{0.07\,\mathrm{J\,m^{-2}}}^{\gamma} \times \overbrace{600\,\mathrm{m^{-1}}}^{k}}{\underbrace{10^3\,\mathrm{kg\,m^{-3}}}_{\rho}} \right)^{1/2} \sim 21\,\mathrm{cm\,s^{-1}}.$$

According to relation between phase and group velocities, the group velocity is 50 percent larger than the phase velocity: $v_g \sim 30\,\mathrm{cm\,s^{-1}}$. This wavelength of 1 cm is roughly the longest wavelength that still qualifies as a ripple, as shown in an earlier figure repeated here:



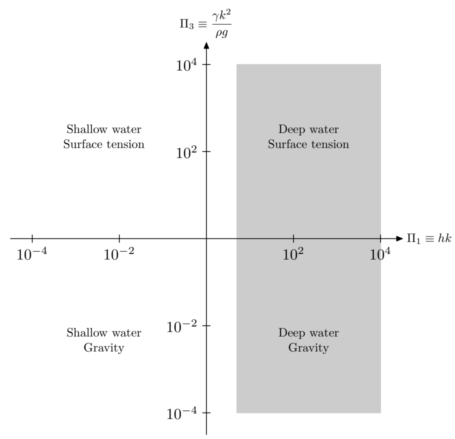
The third dimensionless group, which distinguishes ripples from gravity waves, has value

$$\Pi_3 \equiv \frac{\gamma k^2}{\rho g} \sim \frac{\overbrace{0.07\,\mathrm{J\,m^{-2}}}^{\gamma} \times \overbrace{3.6 \cdot 10^5\,\mathrm{m^{-2}}}^{k^2}}{\underbrace{10^3\,\mathrm{kg\,m^{-3}}}_{\rho} \times \underbrace{10\,\mathrm{m\,s^{-2}}}_{g}} \sim 2.6.$$

With a slightly smaller k, the value of Π_3 would slide into the gray zone $\Pi_3 \approx 1$. If k were yet smaller, the waves would be gravity waves. Other ripples, with a larger k, have a shorter wavelength, and therefore move faster: $21\,\mathrm{cm\,s^{-1}}$ is roughly the minimum phase velocity for ripples. This minimum speed explains why we see mostly $\lambda \sim 1\,\mathrm{cm}$ ripples when we drop a pebble in a pond. The pebble excites ripples of various wavelengths; the shorter ones propagate faster and the 1 cm ones straggle, so we see the stragglers clearly, without admixture of other ripples.

10.1.11 Combining ripples and gravity waves on deep water

With two corners assembled – gravity waves and ripples in deep water – you can connect the corners to form the deep-water edge. The dispersion relations, for convenience restated here, are



$$\omega^2 = \begin{cases} gk, & \text{gravity waves;} \\ \gamma k^3/\rho, & \text{ripples.} \end{cases}$$

With a little courage, you can combine the relations in these two extreme regimes to produce a dispersion relation valid for gravity waves, for ripples, and for waves in between.

Both functional forms came from the same physical argument of balancing kinetic and potential energies. The difference was the source of the potential energy: gravity or surface tension. On the top half of the world of waves, surface tension dominates gravity; on the bottom half, gravity dominates surface tension. Perhaps in the intermediate region, the two contributions to the potential energy simply add. If so, the combination dispersion relation is the sum of the two extremes:

$$\omega^2 = gk + \gamma k^3/\rho.$$

This result is exact (which is why we used an equality). When in doubt, try the simplest solution.

You can increase your confidence in this result by using the effective gravity produced by surface tension. The two sources of gravity – real and effective – simply add, to make

$$g_{\text{total}} = g + g_{\text{surface tension}} = g + \frac{\gamma k^2}{\rho}.$$

Replace g by g_{total} in $\omega^2 = gk$ reproduces the deep-water dispersion relation:

$$\omega^2 = \left(g + \frac{\gamma k^2}{\rho}\right) k = gk + \gamma k^3/\rho.$$

This dispersion relation tells us wave speeds for all wavelengths or wavenumbers. The phase velocity is

$$v_{\text{ph}} \equiv \frac{\omega}{k} = \sqrt{\frac{\gamma k}{\rho} + \frac{g}{k}}.$$

Let's check upstairs and downstairs. Surface tension and gravity drive the waves, so γ and g should be upstairs. Inertia slows the waves, so ρ should be downstairs. The phase velocity passes these tests.

As a function of wavenumber, the two terms in the square root compete to increase the speed. The surface-tension term wins at high wavenumber; the gravity term wins at low wavenumber. So there is an intermediate, minimum-speed wavenumber, k_0, which we can estimate by balancing the surface tension and gravity contributions:

$$\frac{\gamma k_0}{\rho} \sim \frac{g}{k_0}.$$

This computation is an example of order-of-magnitude minimization. The minimum-speed wavenumber is

$$k_0 \sim \sqrt{\frac{\rho g}{\gamma}}.$$

Interestingly, $1/k_0$ is the maximum size of raindrops. At this wavenumber $\Pi_3 = 1$: These waves lie just on the border between ripples and gravity waves. Their phase speed is

$$v_0 \sim \sqrt{\frac{2g}{k_0}} \sim \left(\frac{4\gamma g}{\rho}\right)^{1/4}.$$

In water, the critical wavenumber is $k_0 \sim 4\,\mathrm{cm}^{-1}$, so the critical wavelength is $\lambda_0 \sim 1.5\,\mathrm{cm}$; the speed is

$$v_0 \sim 23\,\mathrm{cm\,s}^{-1}.$$

We derived the speed dishonestly. Instead of using the maximum–minimum methods of calculus, we balanced the two contributions. A calculus derivation confirms the minimum phase velocity. A tedious calculus calculation shows that the minimum group velocity is

$$v_\mathrm{g} \approx 17.7\,\mathrm{cm\,s}^{-1}.$$

[If you try to reproduce this calculation, be careful because the minimum group velocity is not the group velocity at k_0.]

Let's do the minimizations honestly. The calculation is not too messy if it's done with good formula hygiene plus a useful diagram, and the proper method is useful in many physical maximum–minimum problems. We illustrate the methods by finding the minimum of the phase velocity. That equation contains constants – ρ, γ, and g – which carry through all the differentiations. To simplify the manipulations, choose a convenient set of units in which

$$\rho = \gamma = g = 1.$$

The analysis of waves uses three basic dimensions: mass, length, and time. Choosing three constants equal to unity uses up all the freedom. It is equivalent to choosing a canonical mass, length, and time, and thereby making all quantities dimensionless. Don't worry: The constants will return at the end of the minimization.

In addition to constants, the phase velocity also contains a square root. As a first step in formula hygiene, minimize instead v_{ph}^2. In the convenient unit system, it is

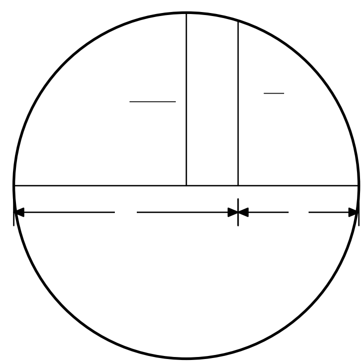
$$v_{\text{ph}}^2 = k + \frac{1}{k}.$$

This minimization does not need calculus, even to do it exactly. The two terms are both positive, so you can use the arithmetic-mean–geometric-mean inequality (affectionately known as AM–GM) for k and $1/k$. The inequality states that, for positive a and b,

$$\underbrace{(a+b)/2}_{AM} \geq \underbrace{\sqrt{ab}}_{GM},$$

with equality when $a = b$.

The figure shows a geometric proof of this inequality. You are invited to convince yourself that the figure is a proof. With $a = k$ and $b = 1/k$ the geometric mean is unity, so the arithmetic mean is ≥ 1. Therefore



$$k + \frac{1}{k} \geq 2,$$

with equality when $k = 1/k$, namely when $k = 1$. At this wavenumber the phase velocity is $\sqrt{2}$. Still in this unit system, the dispersion relation is

$$\omega = \sqrt{k^3 + k},$$

and the group velocity is

$$v_{\text{g}} = \frac{\partial}{\partial k}\sqrt{k^3 + k},$$

which is

$$v_{\text{g}} = \frac{1}{2}\frac{3k^2 + 1}{\sqrt{k^3 + k}}.$$

At $k = 1$ the group velocity is also $\sqrt{2}$: These borderline waves have equal phase and group velocity. This equality is reasonable. In the gravity-wave regime, the phase velocity is greater than the group velocity. In the ripple regime, the phase velocity is less than the group velocity. So they must be equal somewhere in the intermediate regime.

To convert $k = 1$ back to normal units, multiply it by unity in the form of a convenient product of ρ, γ, and g (which are each equal to 1 for the moment). How do you make a length from ρ, γ, and g? The form of the result says that $\sqrt{\rho g/\gamma}$ has units of L^{-1}. So $k = 1$ really means $k = 1 \times \sqrt{\rho g/\gamma}$, which is the same as the order-of-magnitude minimization. This exact calculation shows that the missing dimensionless constant is 1.

The minimum group velocity is more complicated than the minimum phase velocity because it requires yet another derivative. Again, remove the square root and minimize v_{g}^2. The derivative is

$$\frac{\partial}{\partial k} \underbrace{\frac{9k^4 + 6k^2 + 1}{k^3 + k}}_{v_g^2} = \frac{(3k^2 + 1)(3k^4 + 6k^2 - 1)}{(k^3 + k)^2}.$$

Equating this derivative to zero gives $3k^4 + 6k^2 - 1 = 0$, which is a quadratic in k^2, and has positive solution

$$k_1 = \sqrt{-1 + \sqrt{4/3}} \sim 0.393.$$

At this k, the group velocity is

$$v_g(k_1) \approx 1.086.$$

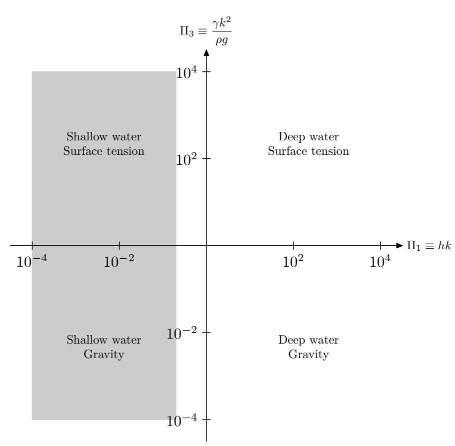
In more usual units, this minimum velocity is

$$v_g \approx 1.086 \left(\frac{\gamma g}{\rho}\right)^{1/4}.$$

With the density and surface tension of water, the minimum group velocity is $17.7\,\mathrm{cm\,s^{-1}}$, as claimed previously.

After dropping a pebble in a pond, you see a still circle surrounding the drop point. Then the circle expands at the minimum group velocity given. Without a handy pond, try the experiment in your kitchen sink: Fill it with water and drop in a coin or a marble. The existence of a minimum phase velocity, is useful for bugs that walk on water. If they move slower than $23\,\mathrm{cm\,s^{-1}}$, they generate no waves, which reduces the energy cost of walking.

10.1.12 Shallow water



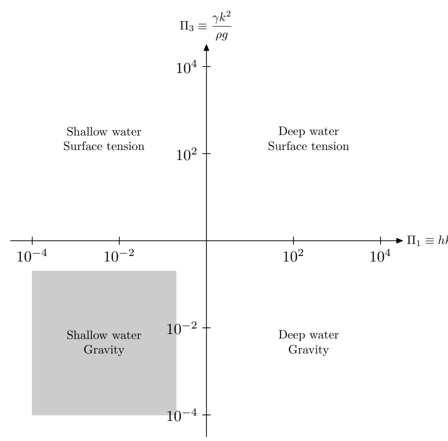
In shallow water, the height h, absent in the deep-water calculations, returns to complicate the set of relevant variables. We are now in the shaded region of the figure. This extra length scale gives too much freedom. Dimensional analysis alone cannot deduce the shallow-water form of the magic function f in the dispersion relation. The slab argument can do the job, but it needs a few modifications for the new physical situation.

In deep water the slab has depth $1/k$. In shallow water, however, where $h \ll 1/k$, the bottom of the ocean arrives before that depth. So the shallow-water slab has depth h. Its length is still $1/k$, and its width is still w. Because the depth changed, the argument about how the water flows is slightly different. In deep water, where the slab has depth equal to length, the slab and surface move the same distance. In shallow water, with a slab thinner by hk, the surface moves more slowly than the slab because less water is being moved around. It moves more slowly by the factor hk. With wave height ξ and frequency ω, the surface moves with velocity $\xi\omega$, so the slab moves (sideways) with velocity $v_{\mathrm{slab}} \sim \xi\omega/hk$. The kinetic energy in the water is contained mostly in the slab, because the upward motion is much slower than the slab motion. This energy is

$$\mathrm{KE}_{\text{shallow}} \sim \underbrace{\rho w h/k}_{mass} \times \underbrace{(\xi\omega/hk)^2}_{v^2} \sim \frac{\rho w \xi^2 \omega^2}{hk^3}.$$

This energy balances the potential energy, a computation we do for the two limiting cases: ripples and gravity waves.

10.1.13 Gravity waves on shallow water



We first specialize to gravity waves – the shaded region in the figure – where water is shallow and wavelengths are long. These conditions include tidal waves,waves generated by undersea earthquakes, and waves approaching a beach. For gravity waves, the potential energy is

$$\mathrm{PE} \sim \rho g w \xi^2/k.$$

This energy came from the distortion of the surface, and it is the same in shallow water (as long as the wave amplitude is small compared with the depth and wavelength). [The dominant force (gravity or surface tension) determines the potential energy. As we see when we study shallow-water ripples, in **Section 10.1.15**, the water depth determines the kinetic energy.]

Balancing this energy against the kinetic energy gives:

$$\underbrace{\frac{\rho w \xi^2 \omega^2}{hk^3}}_{KE} \sim \underbrace{\rho g w \xi^2/k}_{PE}.$$

So

$$\omega^2 = 1 \times ghk^2.$$

Once again, the correct, honestly calculated dimensionless constant (in red) is unity. So, for gravity waves on shallow water, the function f has the form

$$f_{\text{shallow}}(kh, \frac{\gamma k^2}{\rho g}) = kh.$$

Since $\omega \propto k^1$, the group and phase velocities are equal and independent of frequency:

$$v_{\text{ph}} = \frac{\omega}{k} = \sqrt{gh},$$
$$v_{\text{g}} = \frac{\partial\omega}{\partial k} = \sqrt{gh}.$$

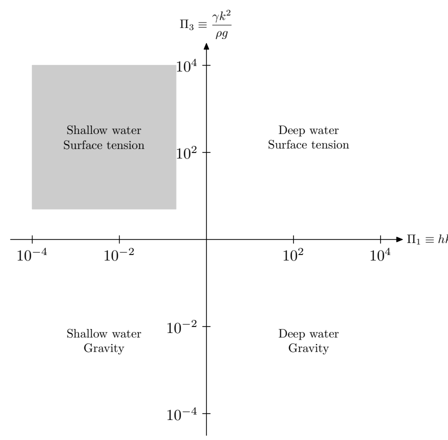
Shallow water is **nondispersive**: All frequencies move at the same velocity, so pulses composed of various frequencies propagate without smearing.

10.1.14 Tidal waves

Undersea earthquakes illustrate the danger in such unity. If an earthquake strikes off the coast of Chile, dropping the seafloor, it generates a shallow-water wave. This wave travels without distortion to Japan. The wave speed is $v \sim \sqrt{4000\,\mathrm{m} \times 10\,\mathrm{m\,s^{-2}}} \sim 200\,\mathrm{m\,s^{-1}}$: The wave can cross a 10^4 km ocean in half a day. As it approaches shore, where the depth decreases, the wave slows, grows in amplitude, and becomes a large, destructive wave hitting land.

10.1.15 Ripples on shallow water

Ripples on shallow water – the shaded region in the figure – are rare. They occur when raindrops land in a shallow rain puddle, one whose depth is less than 1 mm. Even then, only the longest-wavelength ripples, where $\lambda \sim 1$ cm, can feel the bottom of the puddle (the requirement for the wave to be a shallow-water wave). The potential energy of the surface is given by



$$\mathrm{PE}_{\text{ripple}} \sim \gamma \Delta A \sim \gamma w \xi^2 k.$$

Although that formula applied to deep water, the water depth does not affect the potential energy, so we can use the same formula for shallow water.

The dominant force – here, surface tension – determines the potential energy. Balancing the potential energy and the kinetic energy gives:

$$\underbrace{\frac{\rho w \xi^2 \omega^2}{hk^3}}_{KE} \sim \underbrace{\frac{w}{k}\gamma (k\xi)^2}_{PE}.$$

Then

$$\omega^2 \sim \frac{\gamma h k^4}{\rho}.$$

The phase velocity is

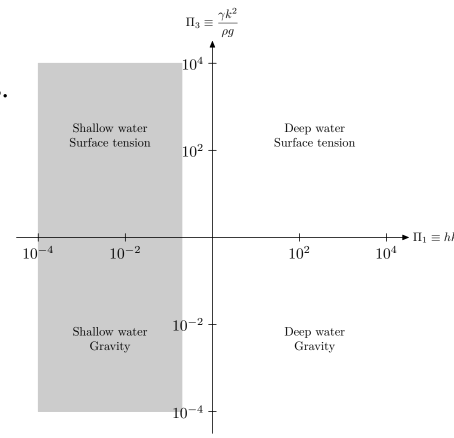
$$v_{\mathrm{ph}} = \frac{\omega}{k} = \sqrt{\frac{\gamma h k^2}{\rho}},$$

and the group velocity is $v_{\mathrm{g}} = 2v_{\mathrm{ph}}$ (the form of the dispersion relation is $\omega \propto k^2$). For $h \sim 1$ mm, this speed is

$$v \sim \left(\frac{0.07\,\mathrm{N\,m^{-1}} \times 10^{-3}\,\mathrm{m} \times 3.6 \cdot 10^{5}\,\mathrm{m^{-2}}}{10^{\mathrm{kg}}\,\mathrm{m^{-3}}} \right)^{1/2} \sim 16\,\mathrm{cm\,s^{-1}}.$$

10.1.16 Combining ripples and gravity waves on shallow water

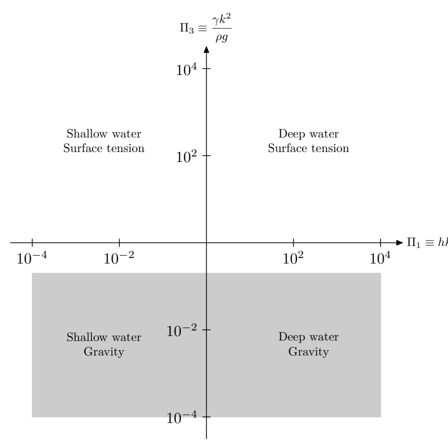
This result finishes the last two corners of the world of waves: shallow-water ripples and gravity waves. Connect the corners to make an edge by studying general shallow-water waves. This region of the world of waves is shaded in the figure. You can combine the dispersion relations for ripples with that for gravity waves using two equivalent methods. Either add the two extreme-case dispersion relations or use the effective gravitational field in the gravity-wave dispersion relation. Either method produces



$$\omega^2 \sim k^2 \left(gh + \frac{\gamma h k^2}{\rho} \right).$$

10.1.17 Combining deep- and shallow-water gravity waves

Now examine the gravity-wave edge of the world, shaded in the figure. The deep- and shallow-water dispersion relations are:



$$\omega^2 = gk \times \begin{cases} 1, & \text{deep water;} \\ hk, & \text{shallow water.} \end{cases}$$

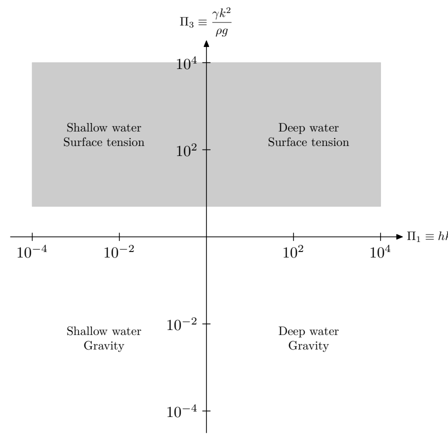
To interpolate between the two regimes requires a function $f(hk)$ that asymptotes to 1 as $hk \to \infty$ and to hk as $hk \to 0$. Arguments based on guessing functional forms have an honored history in physics. Planck derived the blackbody spectrum by interpolating between the high- and low-frequency limits of what was known at the time. We are not deriving quantum mechanics, but the principle is the same: In new areas, whether new to you or new to everyone, you need a bit of courage. One simple interpolating function is $\tanh hk$. Then the one true gravity wave dispersion relation is:

$$\omega^2 = gk \tanh hk.$$

This educated guess is plausible because $\tanh hk$ falls off exponentially as $h \to \infty$, in agreement with the argument based on Laplace's equation. In fact, this guess is correct.

10.1.18 Combining deep- and shallow-water ripples

We now examine the final edge: ripples in shallow and deep water, as shown in the figure. In **Section 10.1.17**, $\tanh kh$ interpolated between hk and 1 as hk went from 0 to ∞ (as the water went from shallow to deep). Probably the same trick works for ripples, because the Laplace-equation argument, which justified the $\tanh kh$, does not depend on the restoring force. The relevant dispersion relations:



$$\omega^2 = \begin{cases} \gamma k^3/\rho, & \text{if } kh \gg 1; \\ \gamma h k^4/\rho, & \text{if } kh \ll 1. \end{cases}$$

If we factor out $\gamma k^3/\rho$, the necessary transformation becomes clear:

$$\omega^2 = \frac{\gamma k^3}{\rho} \times \begin{cases} 1, & \text{if } kh \gg 1; \\ hk, & \text{if } kh \ll 1. \end{cases}$$

This ripple result looks similar to the gravity-wave result, so make the same replacement:

$$\begin{cases} 1, & \text{if } kh \gg 1, \\ hk, & \text{if } kh \ll 1, \end{cases} \quad \text{becomes} \quad \tanh kh.$$

Then you get the general ripple dispersion relation:

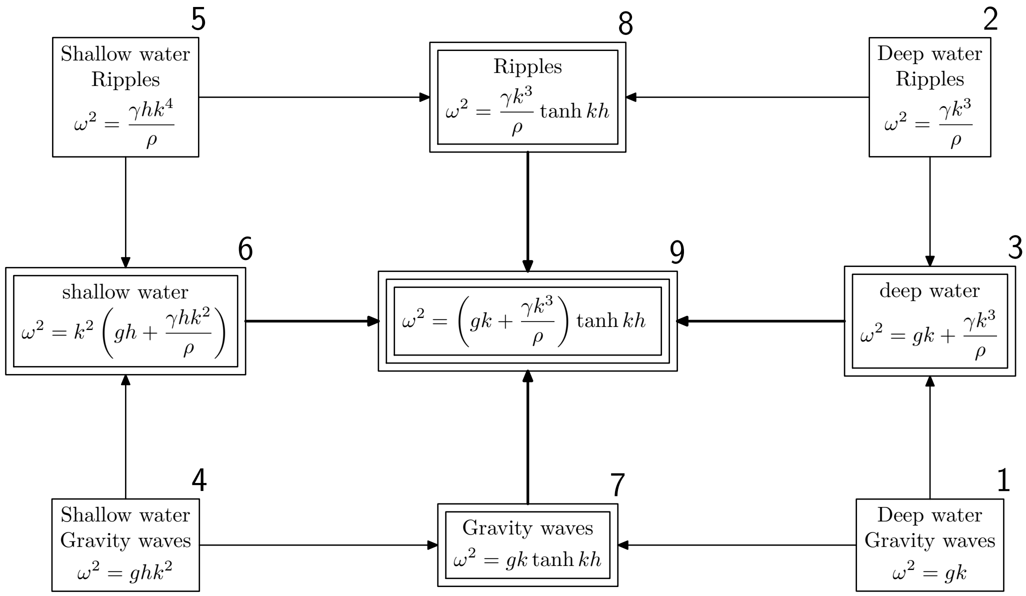
$$\omega^2 = \frac{\gamma k^3}{\rho} \tanh kh.$$

This dispersion relation does not have much practical interest because, at the cost of greater complexity than the deep-water ripple dispersion relation, it adds coverage of only a rare case: ripples on ponds. We include it for completeness, to visit all four edges of the world, in preparation for the grand combination coming up next.

10.1.19 Combining all the analyses

Now we can replace g with g_{total}, to find the One True Dispersion Relation:

$$\omega^2 = (gk + \gamma k^3/\rho) \tanh kh.$$



Each box in the figure represents a special case. The numbers next to the boxes mark the order in which we studied that limit. In the final step (9), we combined all the analyses into the superbox in the center, which contains the dispersion relation for all waves: gravity waves or ripples, shallow water or deep water. The arrows show how we combined smaller, more specialized corner boxes into the more general edge boxes (double ruled), and the edge regions into the universal center box (triple ruled).

In summary, we studied water waves by investigating dispersion relations. We mapped the world of waves, explored the corners and then the edges, and assembled the pieces to form an understanding of the complex, complete solution. The whole puzzle, solved, is shown in the figure. Considering limiting cases and stitching them together makes the analysis tractable and comprehensible.

10.1.20 What you have learned

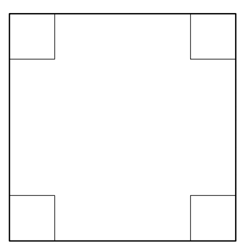
1. *Phase and group velocities.* Phase velocity says how fast crests in a single wave move. In a packet of waves (several waves added together), group velocity is the phase velocity of the envelope.
2. *Discretize.* A complicated functional relationship, such as a dispersion relation, is easier to understand in a discrete limit: for example, one that allows only two (ω, k) combinations. This discretization helped explain the meaning of group velocity.
3. *Four regimes.* The four regimes of wave behavior are characterized by two dimensionless groups: a dimensionless depth and a dimensionless ratio of surface tension to gravitational energy.
4. *Look for springs.* Look for springs when a problem has kinetic- and potential-energy reservoirs and energy oscillates between them. A key characteristic of spring motion is overshoot: The system must zoom past the equilibrium configuration of zero potential energy.

5. *Most missing constants are unity.* In analyses of waves and springs, the missing dimensionless constants are usually unity. This fortunate result comes from the virial theorem, which says that the average potential and kinetic energies are equal for a $F \propto r$ force (a spring force). So balancing the two energies is exact in this case.
6. *Minimum speed.* Objects moving below a certain speed (in deep water) generate no waves. This minimum speed is the result of cooperation between gravity and surface tension. Gravity keeps long-wavelength waves moving quickly. Surface tension keeps short-wavelength waves moving quickly.
7. *Shallow-water gravity waves are non-dispersive.* Gravity waves on shallow water (which includes tidal waves on oceans!) travel at speed $\sqrt{gh}$, independent of wavelength.
8. *Froude number.* The Froude number, a ratio of kinetic to potential energy, determines the maximum speed of speedboats and of walking.

Exercises

AM–GM

Prove the arithmetic mean–geometric mean inequality by another method than the circle in the text. Use AM–GM for the following problem normally done with calculus. You start with a unit square, cut equal squares from each corner, then fold the flaps upwards to make a half-open box. How large should the squares be in order to maximize its volume?



Minima without calculus.

Impossible

How can tidal waves on the ocean (typical depth $\sim 4\,\text{km}$) be considered shallow water?

Oven dish

Partly fill a rectangular glass oven dish with water and play with the waves. Give the dish a slight slap and watch the wave go back and forth. How does the wave speed time vary with depth of water? Does your data agree with the theory in this chapter?

Minimum-wave speed

Take a toothpick and move it through a pan of water. By experiment, find the speed at which no waves are generated. How well does it agree with the theory in this chapter?

Kelvin wedge

Show that the opening angle in a ship wake is $2\sin^{-1}(1/3)$.