# Capacitors and inductors

We continue with our analysis of linear circuits by introducing two new passive and linear elements: the capacitor and the inductor.
All the methods developed so far for the analysis of linear resistive circuits are applicable to circuits that contain capacitors and inductors.

Unlike the resistor which dissipates energy, ideal capacitors and inductors store energy rather than dissipating it.

## Capacitor:

In both digital and analog electronic circuits a capacitor is a fundamental element. It enables the filtering of signals and it provides a fundamental memory element.
The capacitor is an element that stores energy in an electric field.

The circuit symbol and associated electrical variables for the capacitor is shown on Figure 1.

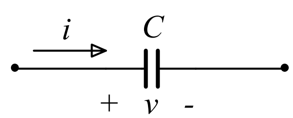



**Figure 1. Circuit symbol for capacitor**

The capacitor may be modeled as two conducting plates separated by a dielectric as shown on Figure 2.

When a voltage $v$ is applied across the plates, a charge $+q$ accumulates on one plate and a charge $-q$ on the other.

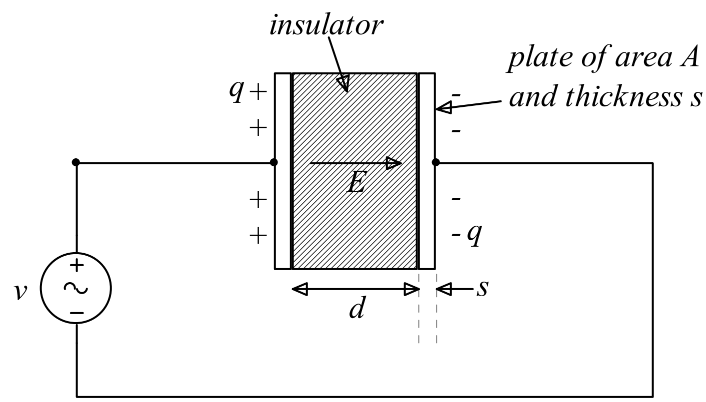



**Figure 2. Capacitor model**

If the plates have an area $A$ and are separated by a distance $d$, the electric field generated across the plates is

$$E = \frac{q}{\varepsilon \mathrm{A}} \tag{1.1}$$

and the voltage across the capacitor plates is

$$v = Ed = \frac{qd}{\varepsilon A} \tag{1.2}$$

The current flowing into the capacitor is the rate of change of the charge across the capacitor plates $i = \frac{dq}{dt}$. And thus we have,

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(\frac{\varepsilon A}{d} v\right) = \frac{\varepsilon A}{d}\frac{dv}{dt} = C\frac{dv}{dt} \tag{1.3}$$

The constant of proportionality $C$ is referred to as the capacitance of the capacitor. It is a function of the geometric characteristics of the capacitor - plate separation ($d$) and plate area ($A$) - and by the permittivity ($\varepsilon$) of the dielectric material between the plates.

$$C = \frac{\varepsilon A}{d} \tag{1.4}$$

Capacitance represents the efficiency of charge storage and it is measured in units of Farads (F).

The current-voltage relationship of a capacitor is

$$\boxed{i = C\frac{dv}{dt}} \tag{1.5}$$

The presence of time in the characteristic equation of the capacitor introduces new and exciting behavior of the circuits that contain them. Note that for DC (constant in time) signals ($\frac{dv}{dt} = 0$) the capacitor acts as an open circuit ($i=0$). Also note the capacitor does not like voltage discontinuities since that would require that the current goes to infinity which is not physically possible.

If we integrate Equation (1.5) over time we have

$$\int_{-\infty}^{t} i dt = \int_{-\infty}^{t} C \frac{dv}{dt} dt \tag{1.6}$$

$$\begin{aligned} v &= \frac{1}{C} \int_{-\infty}^{t} i\, dt \\ &= \frac{1}{C} \int_{0}^{t} i\, dt + v(0) \end{aligned} \tag{1.7}$$

The constant of integration *v(0)* represents the voltage of the capacitor at time *t=0*. The presence of the constant of integration *v(0)* is the reason for the memory properties of the capacitor.

Let's now consider the circuit shown on Figure 3 where a capacitor of capacitance *C* is connected to a time varying voltage source *v(t)*.

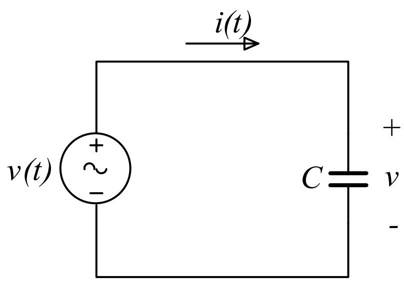



**Figure 3. Fundamental capacitor circuit**

If the voltage *v(t)* has the form

$$v(t) = A\cos(\omega t) \tag{1.8}$$

Then the current *i(t)* becomes

$$\begin{aligned} i(t) &= C\frac{dv}{dt} \\ &= -C\,A\,\omega \sin(\omega t) \\ &= C\,\omega A \cos\left(\omega t + \frac{\pi}{2}\right) \end{aligned} \tag{1.9}$$

Therefore the current going through a capacitor and the voltage across the capacitor are 90 degrees out of phase. It is said that the current **leads** the voltage by 90 degrees.

The general plot of the voltage and current of a capacitor is shown on Figure 4. The current leads the voltage by 90 degrees.

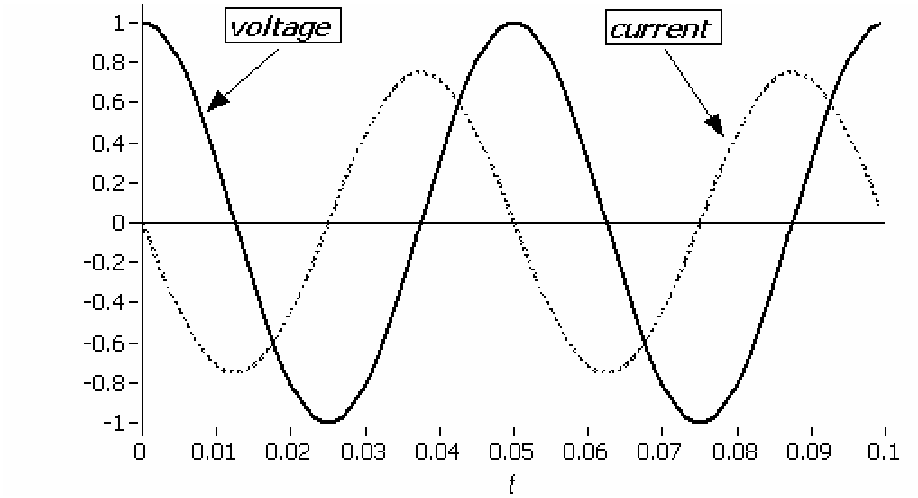



**Figure 4**

If we take the ratio of the peak voltage to the peak current we obtain the quantity

$$Xc = \frac{1}{C\omega} \tag{1.10}$$

*Xc* has the units of Volts/Amperes or Ohms and thus it represents some type of resistance. Note that as the frequency $\omega \to 0$ the quantity *Xc* goes to infinity which implies that the capacitor resembles an open circuit .

**Capacitors do like to pass current at low frequencies**

As the frequency becomes very large $\omega \to \infty$ the quantity Xc goes to zero which implies that the capacitor resembles a short circuit.

**Capacitors like to pass current at high frequencies**

Capacitors connected in series and in parallel combine to an equivalent capacitance. Let's first consider the parallel combination of capacitors as shown on Figure 5. Note that all capacitors have the same voltage, *v*, across them.

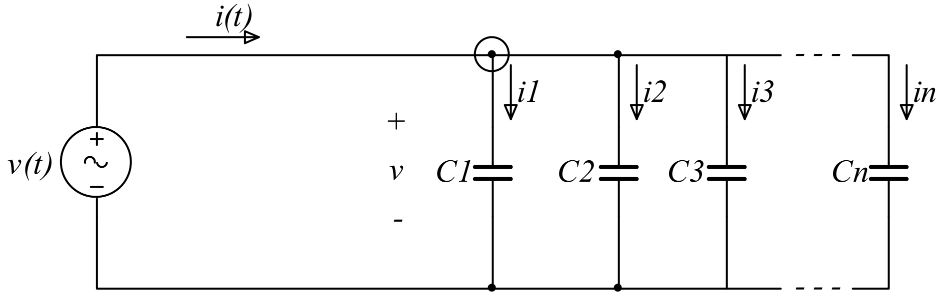



**Figure 5. Parallel combination of capacitors.**

By applying KCL we obtain

$$i = i1 + i2 + i3 + \ldots + in \tag{1.11}$$

And since $ik = Ck\dfrac{dv}{dt}$ we have

$$\begin{aligned} i &= C1\frac{dv}{dt} + C2\frac{dv}{dt} + C3\frac{dv}{dt} + \ldots + Cn\frac{dv}{dt} \\ &= \left(\underbrace{C1 + C2 + C3 + \ldots Cn}_{Ceq}\right)\frac{dv}{dt} \\ &= Ceq\frac{dv}{dt} \end{aligned} \tag{1.12}$$

**Capacitors connected in parallel combine like resistors in series**

Next let's look at the series combination of capacitors as shown on Figure 6.

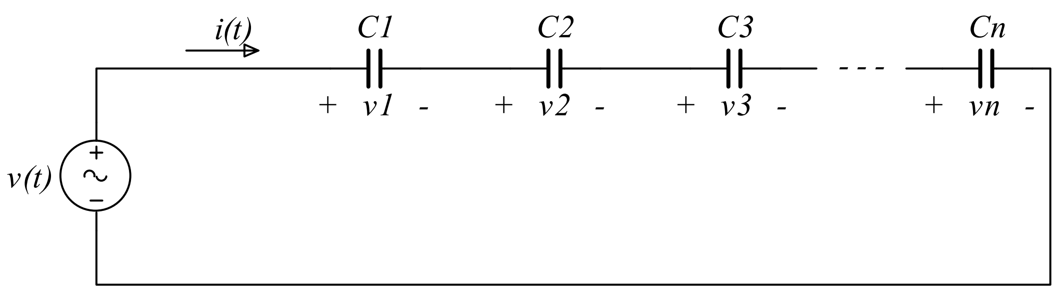



**Figure 6. Series combination of *n* capacitors.**

Now by applying KVL around the loop and using Equation (1.7) we have

$$\begin{aligned} v &= v1 + v2 + v3 + \ldots + vn \\ &= \left(\underbrace{\frac{1}{C1} + \frac{1}{C2} + \frac{1}{C3} + \ldots + \frac{1}{Cn}}_{\frac{1}{Ceq}}\right)\int_0^t i(t)dt + v(0) \\ &= \frac{1}{Ceq}\int_0^t i(t)dt + v(0) \end{aligned} \tag{1.13}$$

**Capacitors in series combine like resistors in parallel**

By extension we can calculate the voltage division rule for capacitors connected in series. Here let's consider the case of only two capacitors connected in series as shown on Figure 7.

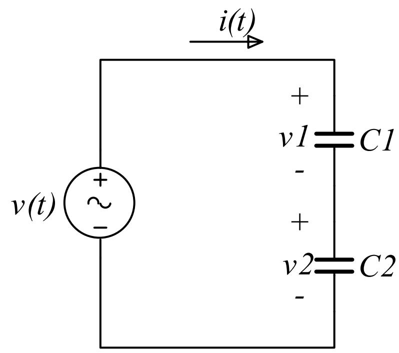



**Figure 7. Series combination of two capacitors**

The same current flows through both capacitors and so the voltages *v1* and *v2* across them are given by:[1]

$$v1 = \frac{1}{C1}\int_0^t idt \tag{1.14}$$

$$v2 = \frac{1}{C2}\int_0^t idt \tag{1.15}$$

And KVL around the loop results in

$$v(t) = \left(\frac{1}{C1} + \frac{1}{C2}\right)\int_0^t idt \tag{1.16}$$

Which in turn gives the voltages *v1* and *v2* in terms of *v* and the capacitances:

$$v1 = v\frac{C2}{C1 + C2} \tag{1.17}$$

$$v2 = v\frac{C1}{C1 + C2} \tag{1.18}$$

Similarly in the parallel arrangement of capacitors (Figure 8) the current division rule is

$$i1 = i\frac{C1}{C1 + C2} \tag{1.19}$$

$$i2 = i\frac{C2}{C1 + C2} \tag{1.20}$$

[1] Assume here that both capacitors are initially uncharged

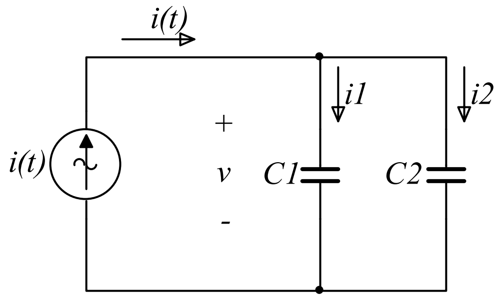



**Figure 8. Parallel arrangement of two capacitors**

The instantaneous power delivered to a capacitor is

$$P(t) = i(t)v(t) \tag{1.21}$$

The energy stored in a capacitor is the integral of the instantaneous power. Assuming that the capacitor had no charge across its plates at $t = -\infty \left[ v(-\infty) = 0 \right]$ then the energy stored in the capacitor at time *t* is

$$\begin{aligned} E(t) &= \int_{-\infty}^{t} P(\tau) d\tau \\ &= \int_{-\infty}^{t} v(\tau) i(\tau)\, d\tau \\ &= \int_{-\infty}^{t} v(\tau) C \frac{dv(\tau)}{d\tau} d\tau \\ &= \frac{1}{2} C\ v(t)^2 \end{aligned} \tag{1.22}$$

Real Capacitors.

If the dielectric material between the plates of a capacitor has a finite resistivity – as compared to infinite resistivity in the case of an ideal capacitor – then there is going to be a small amount of current flowing between the capacitor plates. In addition there are lead resistance and plate effects.

In general the circuit model of a non-ideal capacitor is shown on Figure 9

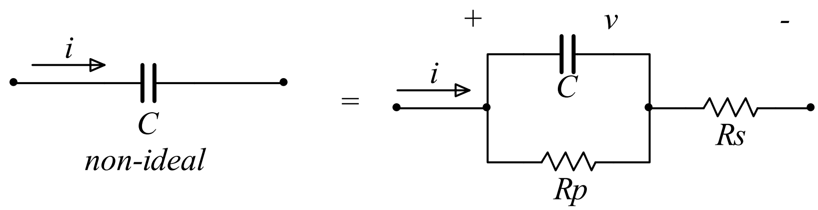



**Figure 9. Circuit of non-ideal capacitor**

The resistance *Rp* is typically very large and it represents the resistance of the dielectric material. Resistance *Rs* is typically small and it corresponds to the lead and plate resistance as well as resistance effects due to the operating conditions (for example signal frequency)

In practice we are concerned with the in series resistance of a capacitor called the Equivalent Series Resistance (ESR). ESR is a very important capacitor characteristic and must be taken into consideration in circuit design. Therefore the non-ideal capacitor model of interest to us is shown on

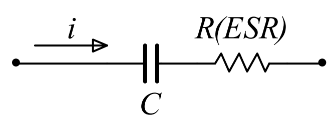



**Figure 10. Non-ideal capacitor with series resistor.**

Typical values of ESR are in the mΩ-Ω range.

A capacitor stores energy in the form of an electric field

Current-voltage relationship $i = C\frac{dv}{dt}, \quad v = \frac{1}{C}\int idt$

In DC the capacitor acts as an open circuit

The capacitance $C$ represents the efficiency of storing charge.

The unit of capacitance is the Farad (F). 1 Farad=1Coulomb/1Volt

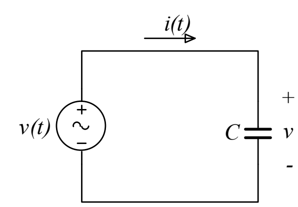
Typical capacitor values are in the mF ($10^{-3}$ F) to pF ($10^{-12}$ F)

The energy stored in a capacitor is $E = \frac{1}{2}Cv^2$

Large capacitors should always be stored with shorted leads.

Example:

A 47μF capacitor is connected to a voltage which varies in time as $v(t) = 20\sin(200\pi t)$ volts.
Calculate the current *i(t)* through the capacitor



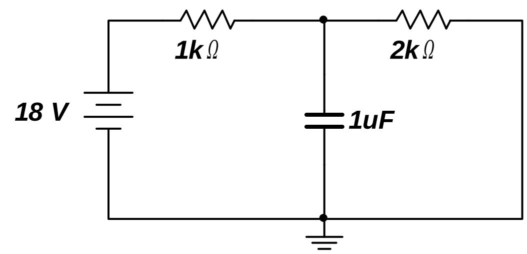

The current is given by

$$i = C\frac{dv}{dt}$$

$$= 47\times10^{-6}\frac{d}{dt}20\sin(200\pi t) = 47\times10^{-6}\times20\times200\pi\cos(200\pi t) = 0.59\cos(200\pi t)\ \mathit{Amperes}$$

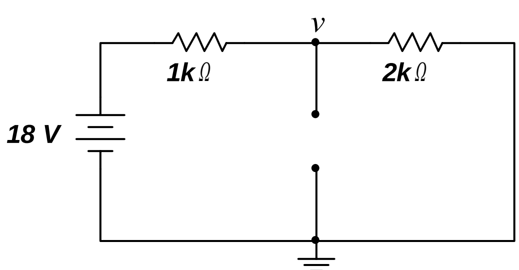
Example:

Calculate the energy stored in the capacitor of the circuit to the right under DC conditions.




In order to calculate the energy stored in the capacitor we must determine the voltage across it and then use Equation (1.22).

We know that under DC conditions the capacitor appears as an open circuit (no current flowing through it). Therefore the corresponding circuit is

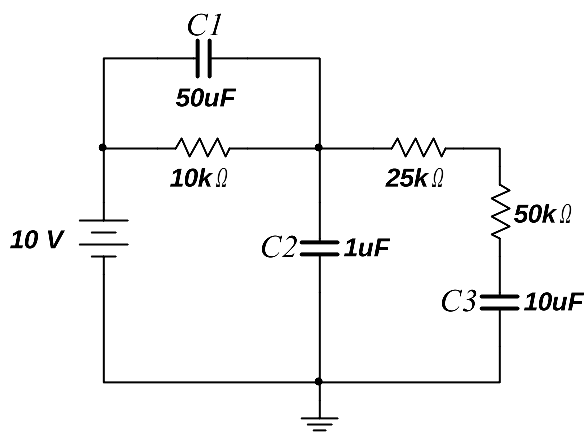



And from the voltage divider formed by the 1kΩ and the 2kΩ resistors the voltage *v* is 12Volts. Therefore the energy stored in the capacitor is

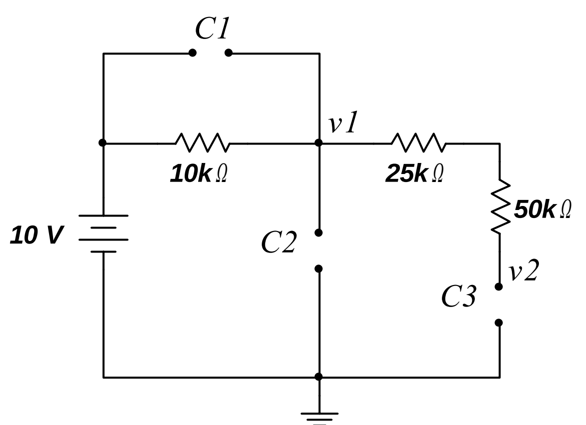
$$Ec = \frac{1}{2}Cv^2 = \frac{1}{2}1\times10^{-6}\times12^2 = 72\,\mu\text{Joules}$$

Example

Calculate the energy stored in the capacitors of the following circuit under DC conditions.




Again DC conditions imply that the capacitor behaves like an open circuit and the corresponding circuit is




From this circuit we see that the voltages v1 and v2 are both equal to 10 Volts and thus the voltage across capacitor C1 is 0 Volts.

Therefore the energy stored in the capacitors is:

For capacitor C1: 0 Joules

For capacitor C2: $E_{C2} = \frac{1}{2} C2 v^2 = \frac{1}{2} 1 \times 10^{-6} \times 10^2 = 50\,\mu\text{Joules}$

For capacitor C3: $E_{C3} = \frac{1}{2} C3 v^2 = \frac{1}{2} 10 \times 10^{-6} \times 10^2 = 500\,\mu\text{Joules}$

**Inductors**

The inductor is a coil which stores energy in the magnetic field

Consider a wire of length *l* forming a loop of area *A* as shown on Figure 11. A current *i(t)* is flowing through the wire as indicated. This current generates a magnetic field *B* which is equal to

$$B(t) = \mu \frac{i(t)}{l} \tag{1.23}$$

Where $\mu$ is the magnetic permeability of the material enclosed by the wire.

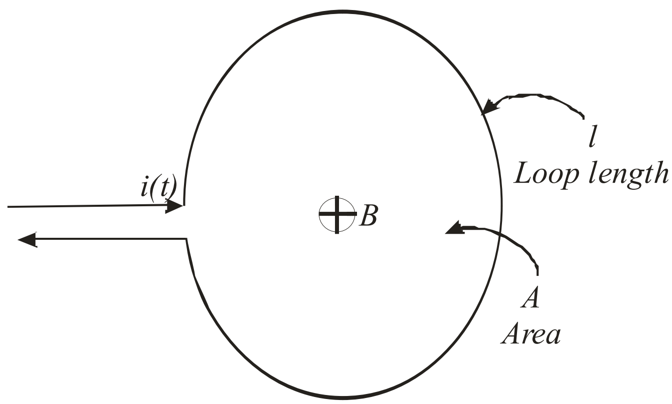



**Figure 11. Current loop for the calculation of inductance**

The magnetic flux, Φ, through the loop of area A is

$$\begin{aligned} \Phi &= AB(t) \\ &= \frac{A\mu}{l} i(t) \\ &= Li(t) \end{aligned} \tag{1.24}$$

Where we have defined $L \equiv \frac{A\mu}{l}$.

From Maxwell's equations we know that

$$\frac{d\Phi}{dt} = v(t) \tag{1.25}$$

$$\frac{d\, Li(t)}{dt} = v(t) \tag{1.26}$$

And by taking L to be a constant we obtain the current-voltage relationship for this loop of wire also called an inductor.

$$\boxed{v = L\frac{di}{dt}} \tag{1.27}$$

The parameter $L$ is called the inductance of the inductor. It has the unit of Henry (H).

The circuit symbol and associated electrical variables for the inductor is shown on Figure 12

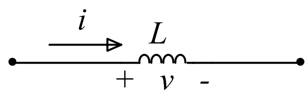



**Figure 12. Circuit symbol of inductor.**

For DC signals ($\frac{di}{dt} = 0$) the inductor acts as a short circuit (*v=0*). Also note the inductor does not like current discontinuities since that would require that the voltage across it goes to infinity which is not physically possible. (We should keep this in mind when we design inductive devices. The current through the inductor must not be allowed to change instantaneously.)

If we integrate Equation (1.27) over time we have

$$\int_{-\infty}^{t} v dt = \int_{-\infty}^{t} L\frac{di}{dt}dt \tag{1.28}$$

$$\begin{aligned} i &= \frac{1}{L}\int_{-\infty}^{t} v\, dt \\ &= \frac{1}{L}\int_{0}^{t} v\, dt + i(0) \end{aligned} \tag{1.29}$$

The constant *i(0)* represents the current through the inductor at time *t=0*. (Note that we have also assumed that the current at $t = -\infty$ was zero.)

Let's now consider the circuit shown on Figure 13 where an inductor of inductance L is connected to a time varying current source *i(t)*.

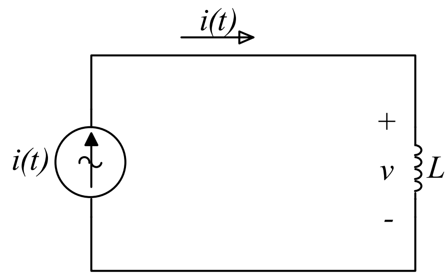



**Figure 13. Fundamental inductor circuit**

If we assume that the current *i(t)* has the form

$$i(t) = I_o \cos(\omega t) \tag{1.30}$$

Then the voltage *v(t)* becomes

$$\begin{aligned} v(t) &= L\frac{di}{dt} \\ &= -L I_o \omega \sin(\omega t) \\ &= L \omega I_o \cos\left(\omega t + \frac{\pi}{2}\right) \end{aligned} \tag{1.31}$$

Therefore the current going through an inductor and the voltage across the inductor are 90 degrees out of phase. Here the voltage **leads** the current by 90 degrees.

The general plot of the voltage and current of an inductor is shown on Figure 14.

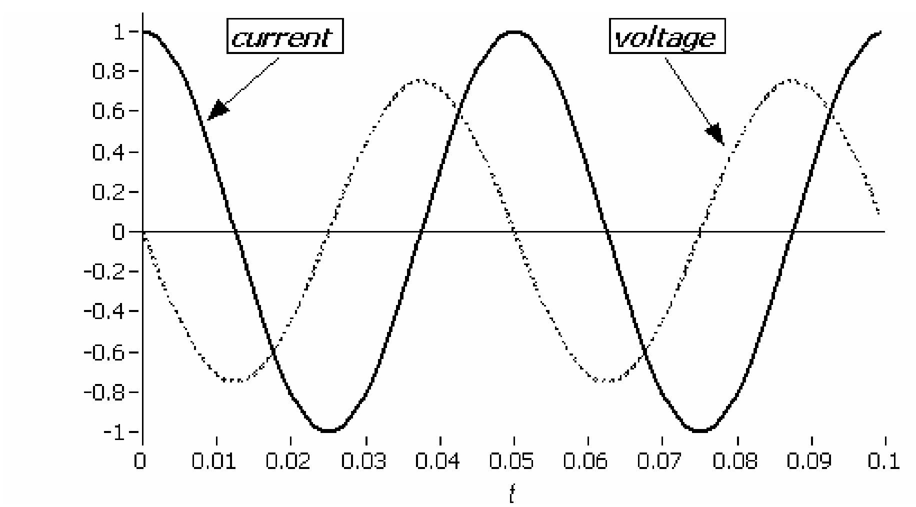



**Figure 14**

Inductor connected in series and in parallel combine to an equivalent inductance. Let's first consider the parallel combination of inductors as shown on Figure 15. Note that all inductors have the same voltage across them.

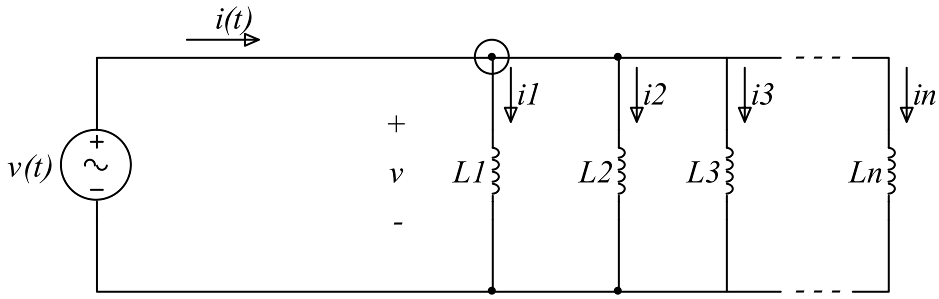



**Figure 15. Parallel combination of inductors.**

By applying KCL we obtain

$$i = i1 + i2 + i3 + \ldots + in \tag{1.32}$$

And since $ik = \frac{1}{Lk}\int_0^t vdt + ik(0)$ we have

$$\begin{aligned} i &= \frac{1}{L1}\int_0^t vdt + i1(0) + \frac{1}{L2}\int_0^t vdt + i2(0) + \frac{1}{L3}\int_0^t vdt + i3(0) + \ldots + \frac{1}{Ln}\int_0^t vdt + in(0) \\ &= \left(\underbrace{\frac{1}{L1} + \frac{1}{L2} + \frac{1}{L3} + \ldots + \frac{1}{Ln}}_{\frac{1}{Leq}}\right)\int_0^t vdt + \underbrace{i1(0) + i2(0) + i3(0) + \ldots + in(0)}_{i(0)} \\ &= \frac{1}{Leq}\int_0^t vdt + i(0) \end{aligned} \tag{1.33}$$

**Inductors in parallel combine like resistors in parallel**

Next let's look at the series combination of inductors as shown on Figure 16.

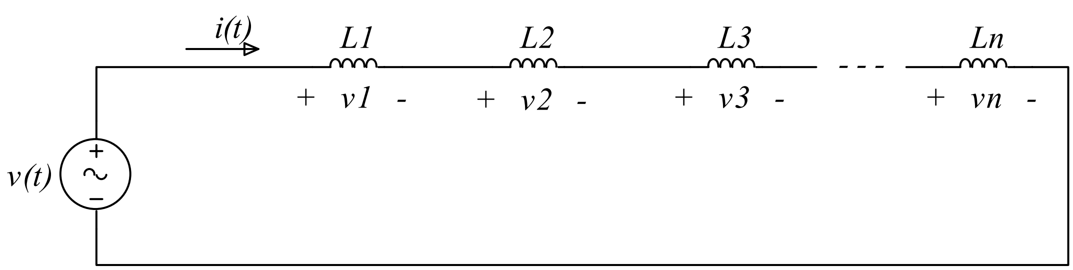



**Figure 16. Series combination of inductors.**

Now by applying KVL around the loop we have

$$\begin{aligned} v &= v1 + v2 + v3 + \ldots + vn \\ &= \left( \underbrace{L1 + L2 + L3 + \ldots + Ln}_{Leq} \right) \frac{di}{dt} \\ &= Leq \frac{di}{dt} \end{aligned} \quad (1.34)$$

**Inductor in series combine like resistor in series**

The energy stored in an inductor is the integral of the instantaneous power delivered to the inductor. Assuming that the inductor had no current flowing through it at $t = -\infty$ $\left[ i(-\infty) = 0 \right]$ then the energy stored in the inductor at time $t$ is

$$\begin{aligned} E(t) &= \int_{-\infty}^{t} P(\tau) d\tau \\ &= \int_{-\infty}^{t} v(\tau) i(\tau)\, d\tau \\ &= \int_{-\infty}^{t} L \frac{di(\tau)}{d\tau} i(\tau) d\tau \\ &= \frac{1}{2} L\ i(t)^2 \end{aligned} \quad (1.35)$$

Real Inductors.

There are two contributions to the non-ideal behavior of inductors.

1. The finite resistance of the wire used to wind the coil
2. The cross turn effects which become important at high frequencies

The non-ideal inductor may thus be modeled as shown on Figure 17

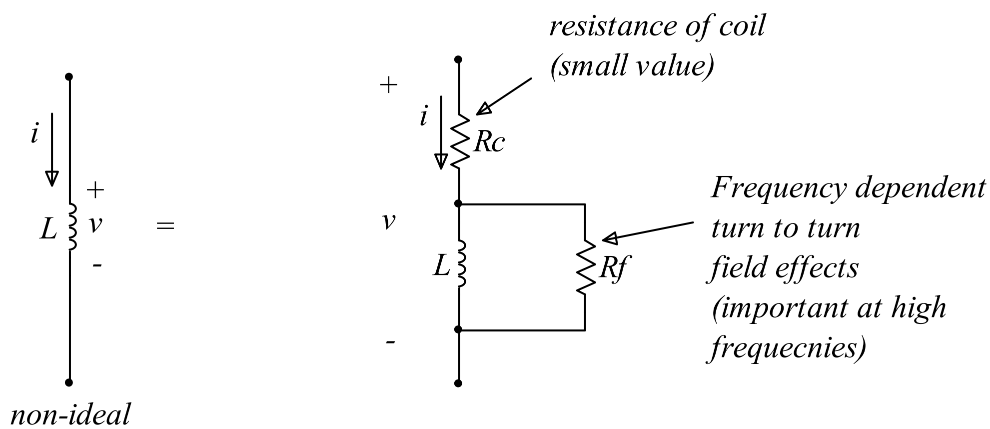



**Figure 17. Circuit momdel of non-ideal inductor**

In addition to the resistive non-idealities of inductors there could also be capacitive effects. These effects usually become important at high frequencies. Unless stated otherwise, these effects will be neglected in out analysis.

A inductor stores energy in a magnetic field

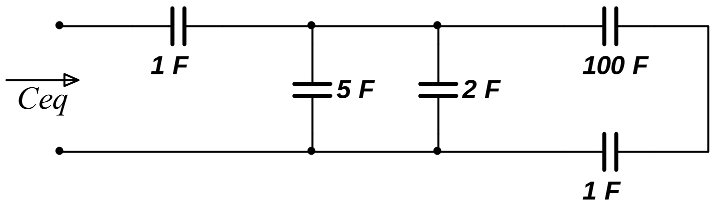
Current-voltage relationship $v = L\frac{di}{dt}, \quad i = \frac{1}{L}\int vdt$

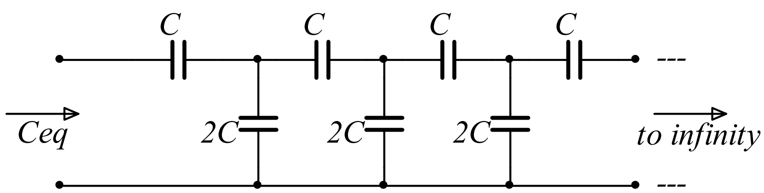
The energy stored in an inductor is $E = \frac{1}{2}Li^2$

In DC the inductor behaves like a short circuit

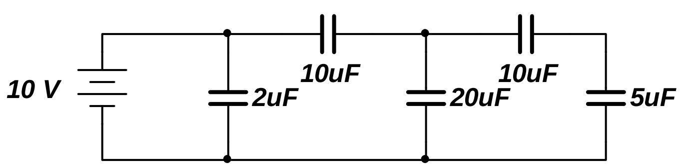
The inductance *L* represents the efficiency of storing magnetic flux.

Problems:
Calculate the equivalent capacitance for the following arrangements:




Calculate the voltage across each capacitor and the energy stored in each capacitor.




In the circuit below the current source provides a current of $i = 10\exp(-2t)$ mA. Calculate the voltage across each capacitor and the energy stored in each capacitor at time *t=2* sec.