Using the Impedance Method

The impedance method allows us to completely eliminate the differential equation approach for the determination of the response of circuits. In fact the impedance method even eliminates the need for the derivation of the system differential equation. Knowledge of the impedance of the various elements in a circuit allows us to apply any of the circuits analysis methods (KVL, KCL, nodal, superposition Thevenin etc.) for the determination of the circuits characteristics: voltages across elements and current through elements.

Before proceeding let's review the impedance definitions and properties of the capacitor and the inductor.

		Frequency (ω) limits	
Element	Impedance	Low $(\omega \to 0)$	High $(\omega \to \infty)$
Capacitor	$Z_C = \dfrac{1}{j\omega C}$	$Z_C \to \infty$ OPEN	$Z_C \to 0$ SHORT
Inductor	$Z_L = j\omega L$	$Z_L \to 0$ SHORT	$Z_L \to \infty$ OPEN

Let's now continue with the analysis of the series RLC circuit shown on Figure 1. We would like to calculate the voltage V_C across the capacitor.

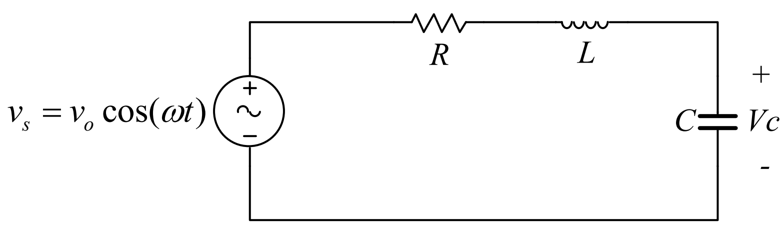


Figure 1. Series RLC circuit

In order to gain a deeper perspective into the power of the impedance method we will first derive the differential equation for V_C and then solve it using the algebraic procedure derived previously.
In turn we will proceed with the application of the impedance method.

The equation for V_C is obtained as follows:

KVL for the circuit mesh gives

$$v_o \cos(\omega t) = i(t)R + L\frac{di(t)}{dt} + V_C \tag{1.1}$$

The current flowing in the circuit is

$$i(t) = C\frac{dV_C}{dt} \tag{1.2}$$

And Equation (1.1) becomes

$$\frac{d^2V_C}{dt^2} + \frac{R}{L}\frac{dV_C}{dt} + \frac{1}{LC}V_C = \frac{v_o}{LC}\cos(\omega t) \tag{1.3}$$

Note that this is a second order differential equation.

For a source term of the form

$$v_o e^{j(\omega t)} \tag{1.4}$$

The solution is

$$V_C(t) = Ae^{j(\omega t+\phi)} \tag{1.5}$$

Substituting into Equation (1.3) we obtain

$$\mathrm{A}\left(-\omega^2 + \frac{R}{L}j\omega + \frac{1}{LC}\right)e^{j\phi} = \frac{v_o}{LC} \tag{1.6}$$

$$\begin{aligned}\mathrm{A}e^{j\phi} &= \frac{1}{\left(-\omega^2 + \frac{R}{L}j\omega + \frac{1}{LC}\right)}\frac{v_o}{LC} \\ &= \frac{v_o}{1-\omega^2 LC + j\omega RC}\end{aligned} \tag{1.7}$$

Which may be simplified as follows

$$\mathrm{A}e^{j\phi} = \frac{v_o}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}}e^{j\left(\tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right)\right)} \tag{1.8}$$

Therefore the amplitude A of V_C is

$$\mathrm{A} = \frac{v_o}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \tag{1.9}$$

And the phase is

$$\phi = \tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right) \tag{1.10}$$

Now we will calculate the voltage V_C by using the impedance method.

In terms of the impedance the RLC circuit is

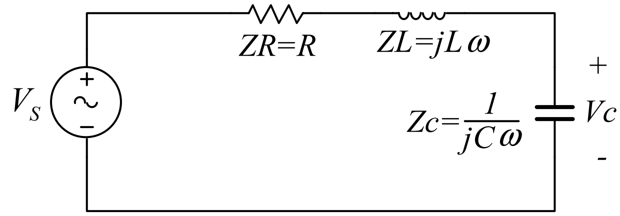


Figure 2

This is now a representation in the frequency domain since impedance is a frequency domain complex quantity

The voltage V_C may now be determined by applying the standard voltage divider relation

$$\begin{aligned} V_C &= Vs\frac{Z_C}{Z_C + Z_L + Z_R} \\ &= Vs\frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L + R} \\ &= Vs\frac{1}{1-\omega^2 LC + j\omega RC} \end{aligned} \tag{1.11}$$

Which is the same as Equation (1.7). Note that we never had to write down the differential equation.

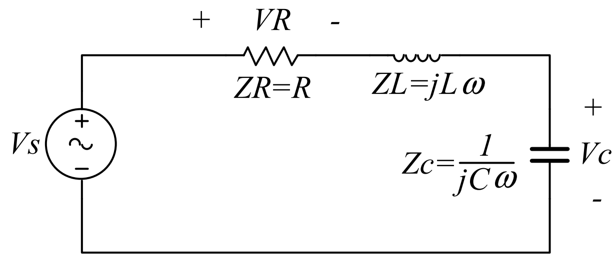
We may now complete the solution by writing $V_C = Ae^{j(\omega t+\phi)}$ and $V_S = v_o e^{j(\omega t)}$ which again gives

$$\mathrm{A} = \frac{v_o}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \tag{1.12}$$

And

$$\phi = \tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right) \tag{1.13}$$

Similarly we can calculate the voltage V_R across resistor R



The voltage divider relationship gives

$$\begin{aligned} V_R &= Vs\frac{Z_R}{Z_C + Z_L + Z_R} \\ &= Vs\frac{R}{\dfrac{1}{j\omega C} + j\omega L + R} \\ &= Vs\frac{j\omega RC}{1-\omega^2 LC + j\omega RC} \end{aligned} \tag{1.14}$$

Upon simplification it becomes

$$\boxed{V_R = Vs\frac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} e^{j\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right)\right)}} \tag{1.15}$$

Note the π/2 phase difference between *VR* and *Vc*.

Also, the voltage across the inductor becomes:

$$\boxed{V_L = Vs\frac{\omega^2 LC}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} e^{j\left(\tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right)\right)}} \tag{1.16}$$

Example: A frequency independent voltage divider

Consider the voltage divider shown below for which the load may be modeled as a parallel combination of resistor *R2* and inductor *L2*.

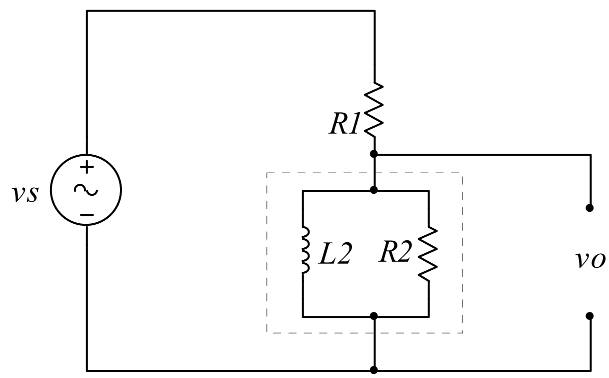


Figure 3

In terms of the impedance the circuit becomes

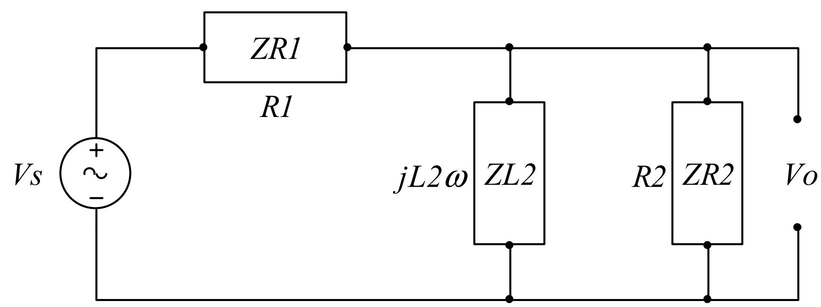


Figure 4

The voltage *Vo* is given by

$$
\begin{aligned}
Vo &= \frac{ZL2 \,//\, ZR2}{ZL2 \,//\, ZR2 + ZR1} \\
&= \frac{ZL2\ ZR2}{ZL2\ ZR2 + ZR1\left(ZL2 + ZR2\right)} \\
&= \frac{j\omega L2R2}{j\omega L2R2 + \left(R1\left(j\omega L2 + R2\right)\right)} \\
&= \frac{j\omega L2R2}{R1R2 + j\omega L2\left(R1 + R2\right)}
\end{aligned} \qquad (1.17)
$$

Equation (1.17) may also be written in polar form as follows

$$
\begin{aligned}
Vo &= \frac{\omega L2 R2}{\sqrt{R1^2 R2^2 + \omega^2 L2^2 \left(R1+R2\right)^2}} e^{j\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L2(R1+R2)}{R1R2}\right)\right)} \\
&= \frac{R2}{R1+R2} \frac{\omega\tau}{\sqrt{1+\omega^2\tau^2}} e^{j\left(\frac{\pi}{2}-\phi\right)}
\end{aligned}
\tag{1.18}
$$

Where

$$
\tau \equiv \frac{L2(R1+R2)}{R1R2} \tag{1.19}
$$

And

$$
\phi = \tan^{-1}\left(\frac{\omega L2(R1+R2)}{R1R2}\right) = \tan^{-1}(\omega\tau) \tag{1.20}
$$

The frequency dependence of the voltage divider is shown on Figure 5. Here we have plotted the amplitude of $\frac{Vo}{Vs}$ as a function of $\omega\tau$ for *R1=R2.* Note the asymptotic value indicated by the dotted line. At high frequencies, for which the inductor acts like an open circuit, the divider ratio reduces to that of the two resistors which in this case is ½ since both resistors are equal. At low frequencies, the low impedance of the inductor reduces the output voltage. At dc ($\omega = 0$) the inductor acts like a short circuit and so *Vo=0*.

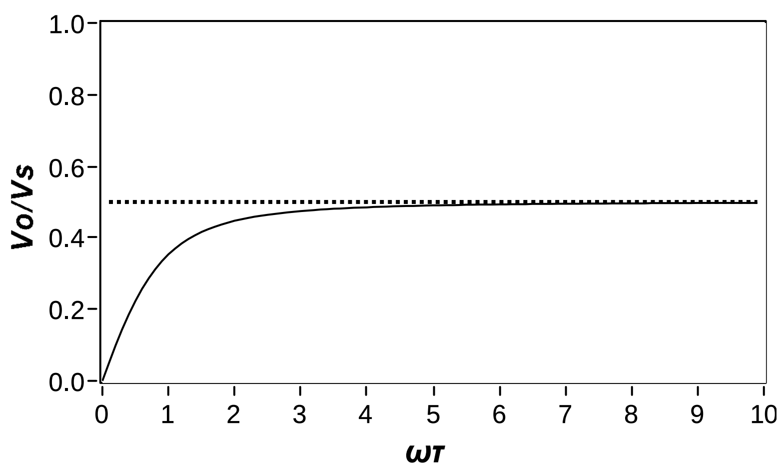
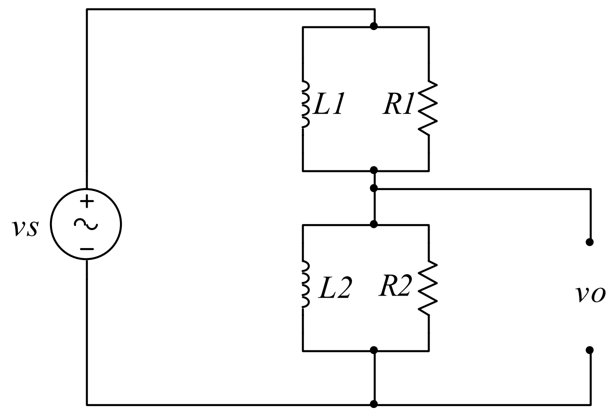


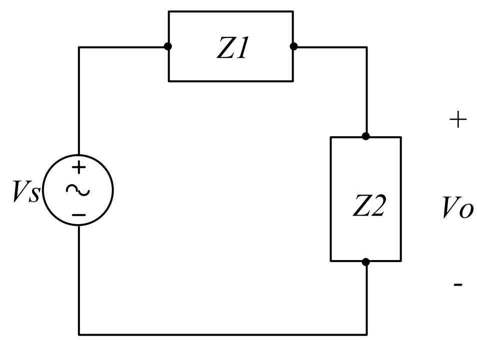
Figure 5

We would like to alter the design of the voltage divider so that it becomes independent of frequency for all frequencies.

One way to address this problem is to add a compensating inductor *L1* as shown on the following schematic.



The equivalent circuit in terms of impedance is



And the voltage divider ratio becomes

$$Vo = Vs\frac{Z2}{Z1+Z2} = Vs\frac{1}{1+\dfrac{Z1}{Z2}} \tag{1.21}$$

Frequency independence implies that the ratio of impedances $\frac{Z1}{Z2}$ must be independent of frequency. This ratio is given by

$$\begin{aligned}\frac{Z1}{Z2} &= \frac{\dfrac{j\omega L1R1}{R1+j\omega L1}}{\dfrac{j\omega L2R2}{R2+j\omega L2}} \\ &= \frac{L1R1}{L2R2}\frac{R2+j\omega L2}{R1+j\omega L1}\end{aligned} \tag{1.22}$$

Equation (1.22) becomes independent of ω if

$$L1 = \frac{R1}{R2} L2 \tag{1.23}$$

Which results in a voltage divider ratio of

$$\frac{Vo}{Vs} = \frac{R2}{R1 + R2} \tag{1.24}$$

A close look at frequency response. (Frequency selection)

As we have discussed previously, the frequency response of a circuit or a system refers to the change in the system characteristics with frequency.

A convenient way to represent this response is to plot the ratio of the response signal to the source signal. For the generic representation shown on Figure 6, the response may be given as the ratio of the output *Y(ω)* to the input *X(ω)*. This ratio is called the transfer function of the system and it is labeled *H(ω)*

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \tag{1.25}$$

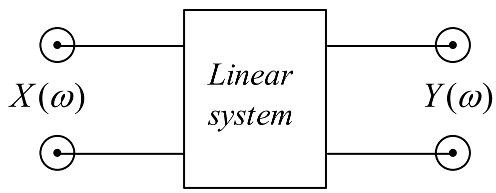


Figure 6

The output and input (*Y(ω)* and *X(ω)*) may represent the amplitude or the phase of the signals. As an example let's consider the RC circuit shown on Figure 7.

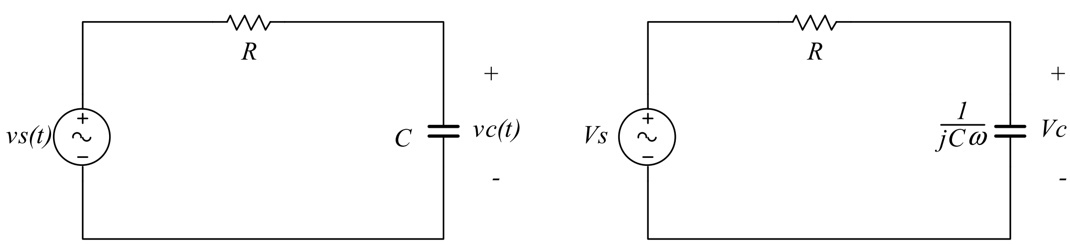


Figure 7

The transfer function for this circuit is

$$H(\omega) = \frac{ZC}{ZR + ZC} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tag{1.26}$$

$$= \frac{1}{1 + j\omega RC}$$

The magnitude and the phase of *H(ω)* are

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}} \tag{1.27}$$

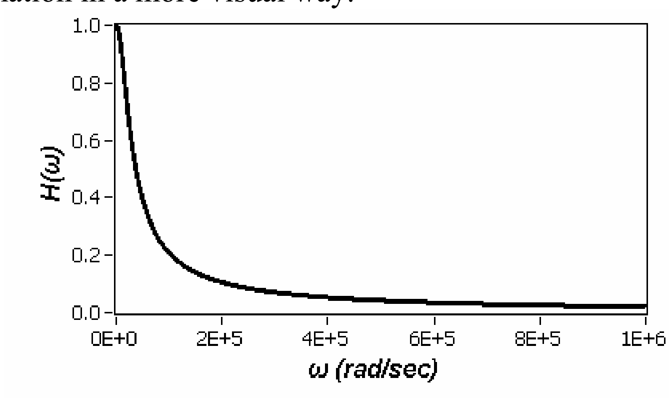
$$\phi = \tan^{-1}(\omega RC) \tag{1.28}$$

In practice the range of frequencies that is used in plotting $|H(\omega)|$ is very wide and thus a linear scale for the frequency axis is often not suitable. In practice $|H(\omega)|$ is plotted versus the logarithm of the frequency.
In addition it is common to plot the transfer function in dB, where

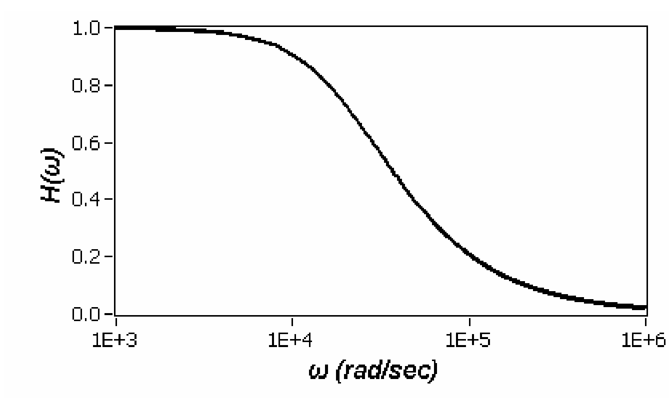
$$|H(\omega)|_{dB} = 20\log_{10}|H(\omega)| \tag{1.29}$$

The plot of $|H(\omega)|_{dB}$ versus $\log(\omega)$ is called the Bode plot.

For our example RC circuit with R=10kΩ and C=47nF Figure 8(a) and (b) show the plot of $|H(\omega)|$ versus ω and $\log(\omega)$ respectively. Note that the semi logarithmic plot presents the information in a more visual way.



(a)



(b)

Figure 8

When $|H(\omega)|$ is calculated in dB the plot versus the logarithm of frequency is shown on Figure 9.

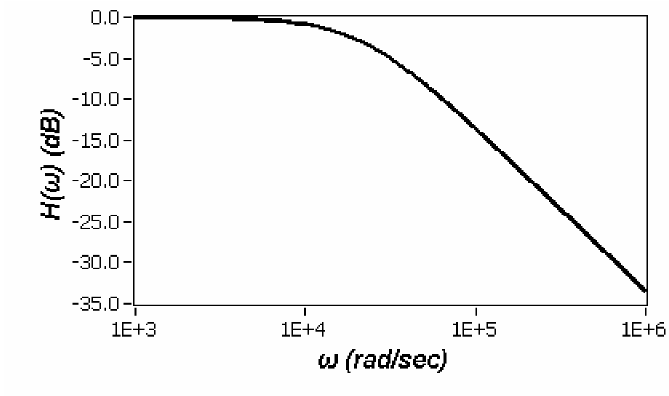


Figure 9

From the above plot we see the strong dependence of the magnitude of the output signal on the frequency.

Figure 10 shows the plot of the phase as a function of frequency.

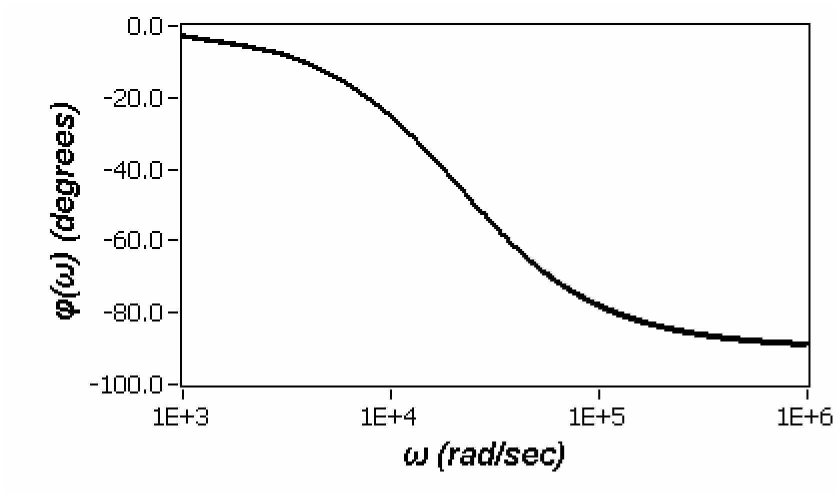
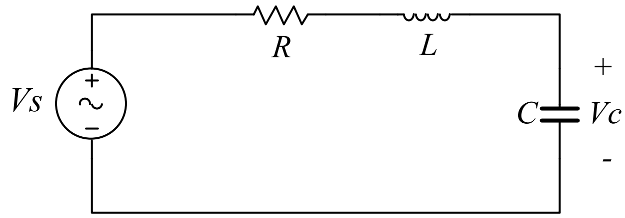


Figure 10

At low ω for which the capacitor acts like an open circuit the phase is zero. At high frequencies the capacitor acts like a short circuit ad the phase goes to -90^{o}.

Now let's continue by graphically exploring the response of RLC circuits.



The amplitude and phase of Vc are given by Equations (1.12) and (1.13) which we rewrite here for convenience.

$$\left|\frac{Vc}{Vs}\right| = |H(\omega)| = \frac{1}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \tag{1.30}$$

$$\phi = \tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right) \tag{1.31}$$

Figure 11 shows the plot of $|H(\omega)|$ as a function of frequency for $R=300\Omega$, $L=47mH$ and $C=47nF$ (the values we also used in laboratory).

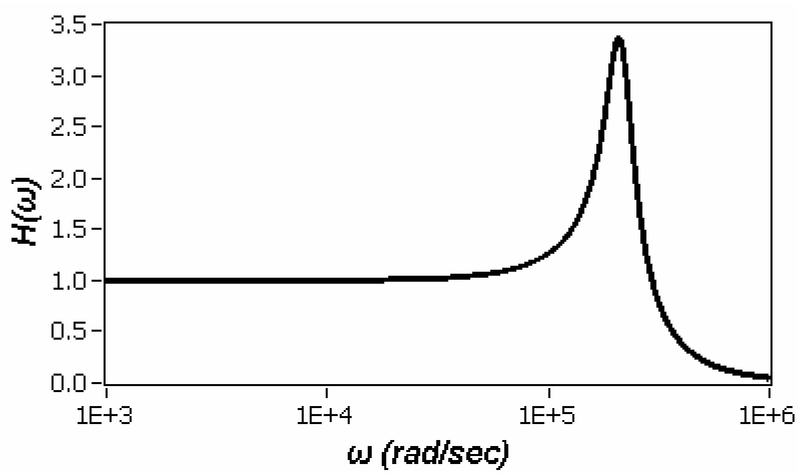


Figure 11

In the limit as $\omega \to 0$, $|H(\omega)| \to 1$. Note also that there is a peak at a certain frequency which by inspection of Equation (1.30) occurs when $1-\omega^2 LC = 0$.

By increasing the value of the resistor R the peak becomes less pronounced. Figure 12 shows the transfer function for $R=300\Omega$ and $R=1.5k\Omega$.

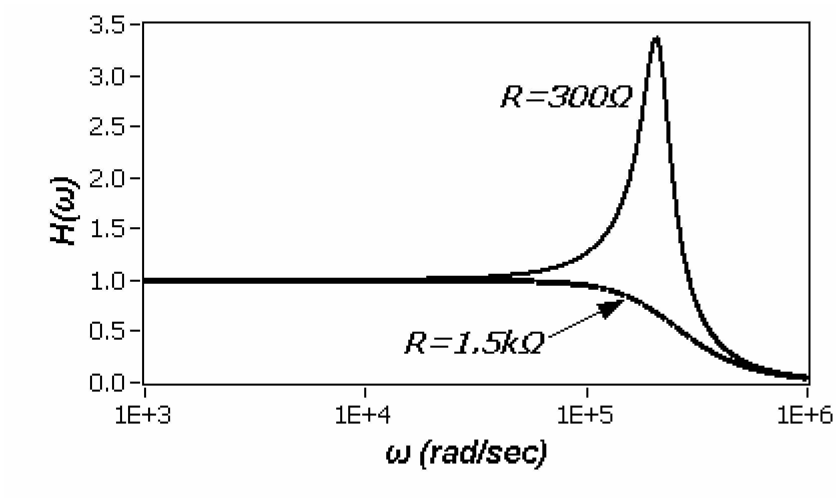


Figure 12

The phase plot is shown on Figure 13. Note that the transition happens again when $1-\omega^2 LC=0$

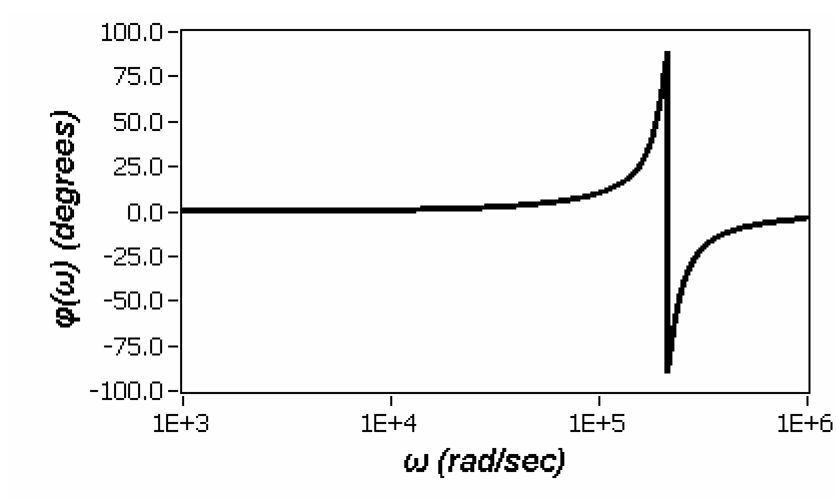
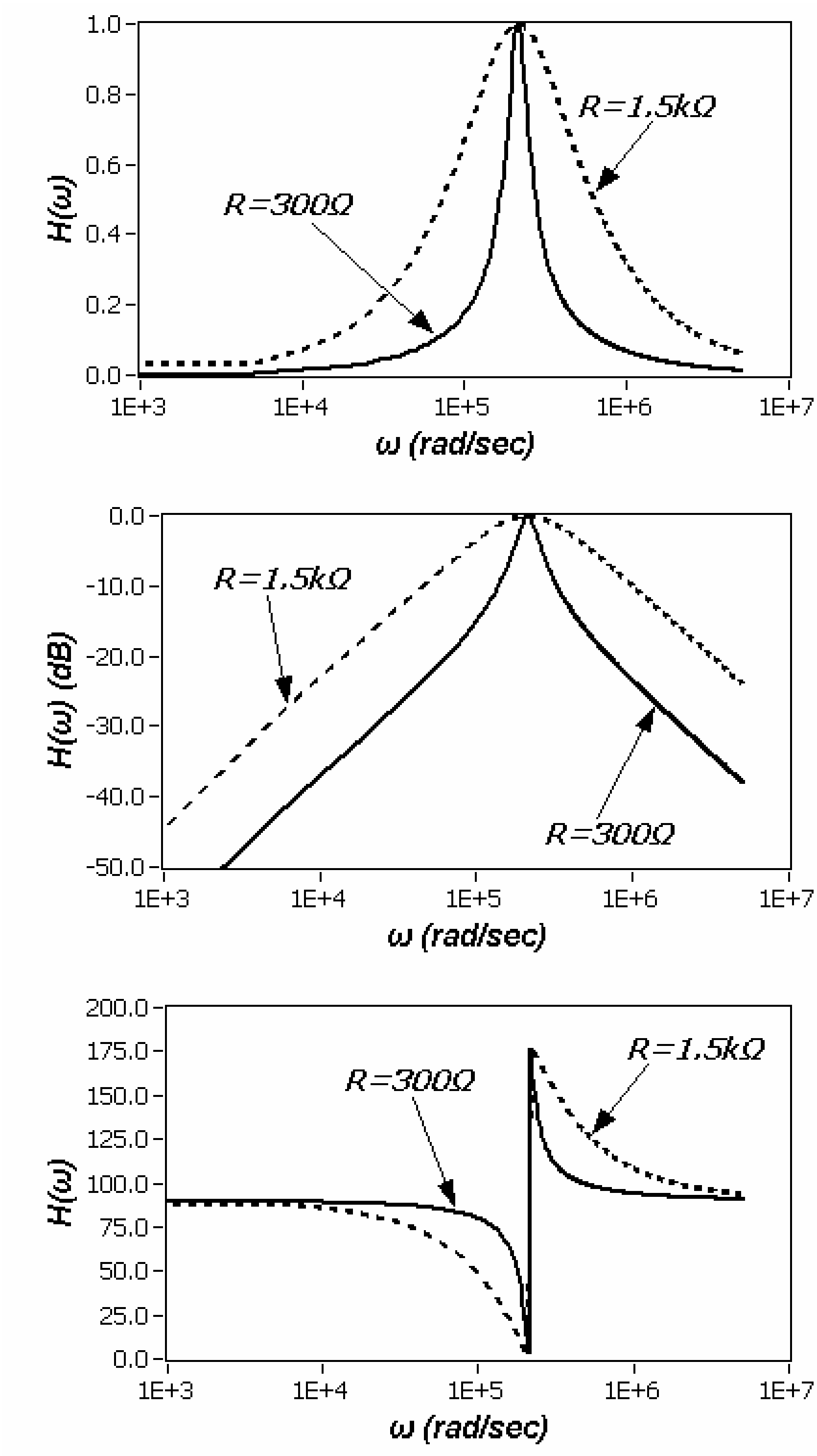
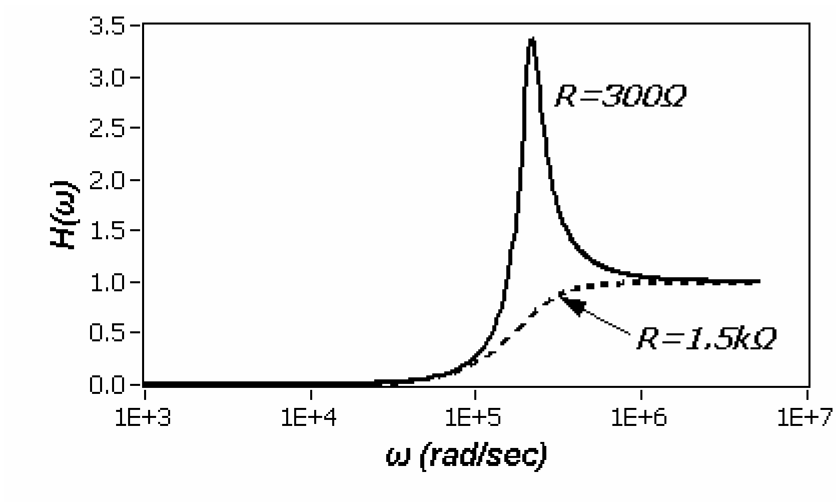


Figure 13

The following Plots show the normalized transfer function for VR and the corresponding phase.



Similarly the normalized transfer function for VL is



In the next two classes we will explore this behavior further and develop their physical significance with regard to their frequency selectivity characteristics.