Sinusoidal Steady State Response:
Frequency domain representation
Impedance

Using the complex forcing function

Let's consider the RL circuit shown on Figure 1. The circuit is driven by the sinusoidal source of the form $v_s(t) = v_o \cos(\omega t)$.

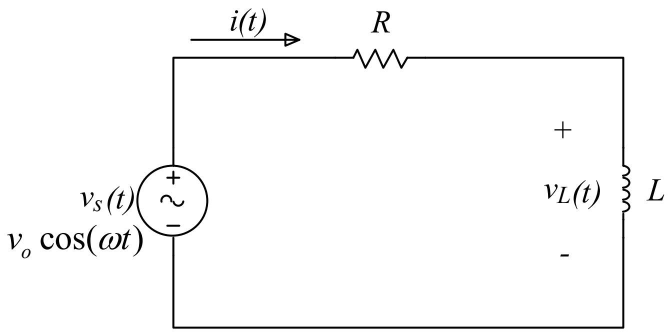


Figure 1

The equation characterizing the system is

$$L\frac{di(t)}{dt} + R\,i(t) = v_o \cos(\omega t) \tag{1.1}$$

By using Euler's identity we know that

$$v_o \cos(\omega t) = \mathrm{Re}\left\{v_o e^{j\omega t}\right\} \tag{1.2}$$

We may use in place of the source function $v_o \cos(\omega t)$ the complex source $v_o e^{j\omega t}$.
This complex function $v_o e^{j\omega t}$ contains the term $v_o \cos(\omega t)$ which is our source function, and the term $j\,v_o \sin(\omega t)$.

Therefore we will proceed with the analysis using the complex function, $v_o e^{j\omega t}$, as the source.

Figure 2 shows the same circuit but with the complex source in the place of the source. (Note that we have used different variables to indicate the response of the circuit to the complex forcing function.)

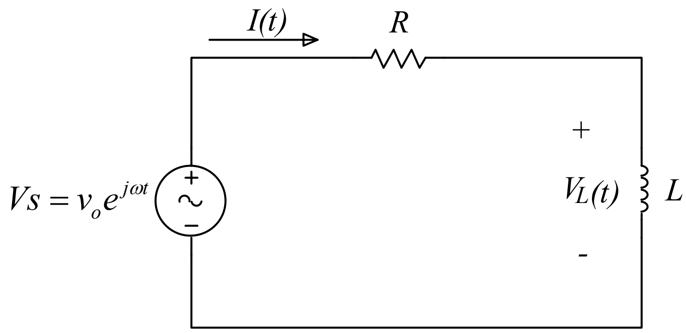


Figure 2

The corresponding complex response for the current *I*(*t*) is

$$I(t) = I_o e^{j(\omega t + \phi)} \tag{1.3}$$

and by substituting the complex form for V_s and *I(t)* into Equation (1.1) we have

$$L\frac{d}{dt} I_o e^{j(\omega t + \phi)} + I_o e^{j(\omega t + \phi)} = v_o e^{j\omega t} \tag{1.4}$$

which upon simplification becomes

$$I_o e^{j\phi}\left(R + j\omega L\right) = v_o \tag{1.5}$$

Equation (1.5) contains the information for both the amplitude I_o and the phase ϕ. By rearranging Equation (1.5) we have

$$I_o e^{j\phi} = \frac{v_o}{R + j\omega L} \tag{1.6}$$

In order to determine I_o and ϕ we express the right hand side of Equation (1.6) in polar coordinates

$$\begin{aligned} I_o e^{j\phi} &= \frac{v_o}{\sqrt{R^2 + \omega^2 L^2}\, e^{j\left(\tan^{-1}\left(\frac{\omega L}{R}\right)\right)}} \\ &= \frac{v_o}{\sqrt{R^2 + \omega^2 L^2}} e^{j\left(-\tan^{-1}\left(\frac{\omega L}{R}\right)\right)} \end{aligned} \tag{1.7}$$

And thus we have an expression for the amplitude I_o and the phase ϕ.

$$I_o = \frac{v_o / R}{\sqrt{1 + \omega^2 \frac{L^2}{R^2}}} \tag{1.8}$$

$$\phi = -\tan^{-1}\left(\frac{\omega L}{R}\right) \tag{1.9}$$

Therefore, the complex response (Equation (1.3)) becomes

$$i(t) = \frac{v_o / R}{\sqrt{1 + \frac{\omega^2 L^2}{R^2}}} \exp\left[\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right] \tag{1.10}$$

which by using Euler's identity is

$$I(t) = \underbrace{\frac{v_o / R}{\sqrt{1 + \frac{\omega^2 L^2}{R^2}}} \cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)}_{\text{Real part of the solution}} + \underbrace{j \frac{v_o / R}{\sqrt{1 + \frac{\omega^2 L^2}{R^2}}} \sin\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)}_{\text{imaginary part of the solution}} \tag{1.11}$$

Since our original forcing source was the cosine function (see Figure 1), the solution to our original problem corresponds to the real part of the complex response function given by Equation (1.11) which is:

$$i(t) = \frac{v_o / R}{\sqrt{1 + \frac{\omega^2 L^2}{R^2}}} \cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right) \tag{1.12}$$

Solution procedure:

1. We would like to find the response of a linear system to a source term of the form $A\cos\omega t$ or $A\sin\omega t$

2. Determine the equation that describes the system

3. Assume a complex source term of the form $Ae^{j\omega t}$

4. Response is then of the form $Be^{j(\omega t+\phi)}$

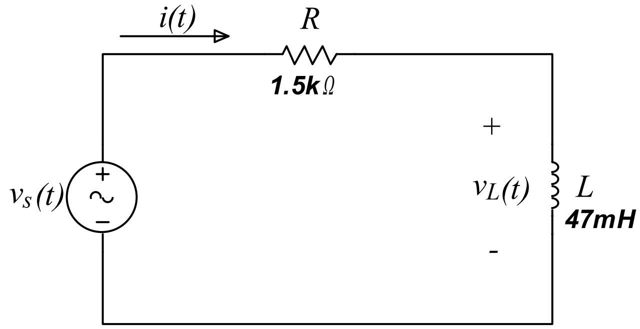
5. Substitute into the system equation and solve for B and φ

6. The response to our original system corresponds to:

$\mathrm{Re}\left\{Be^{j(\omega t+\phi)}\right\}$: If the original source is of the form $A\cos\omega t$

$\mathrm{Im}\left\{Be^{j(\omega t+\phi)}\right\}$: If the original source is of the form $A\sin\omega t$

Example: Let's calculate the voltage $v_L(t)$ for the following circuit for which the source is $v_s(t) = 5\sin(\omega t)$ Volts



The equation that characterizes this circuit is obtained by the application of KVL around the mesh and it is

$$L\frac{di(t)}{dt} + L\,i(t) = v_o \sin(\omega t)$$

Where $L=47mH$, $R=1.5k\Omega$ and $v_o = 5V$.

We will proceed by assuming a complex forcing function of the form

$$V_s(t) = 5e^{j\omega t} \text{ Volts}$$

The corresponding response for the current will be given by

$$I(t) = I_o e^{j(\omega t+\phi)}$$

And the solution for our system is simply the imaginary part of the above expression which is given by

$$i(t) = \frac{v_o / R}{\sqrt{1+\frac{\omega^2 L^2}{R^2}}} \sin\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

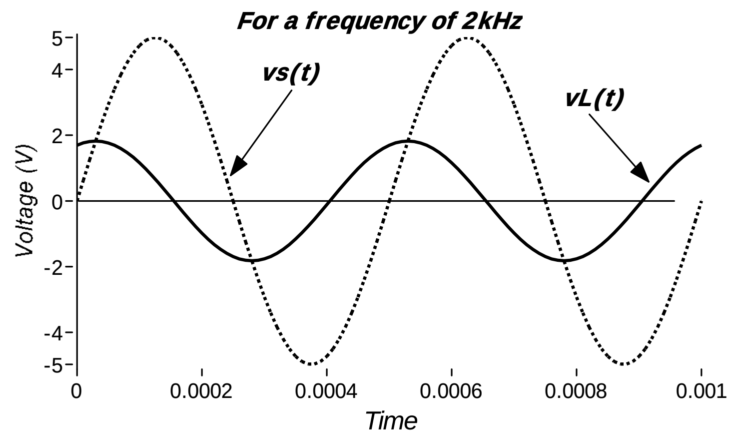
For $L=47mH$, $R=1.5k\Omega$ and $v_o = 5V$ the solution becomes

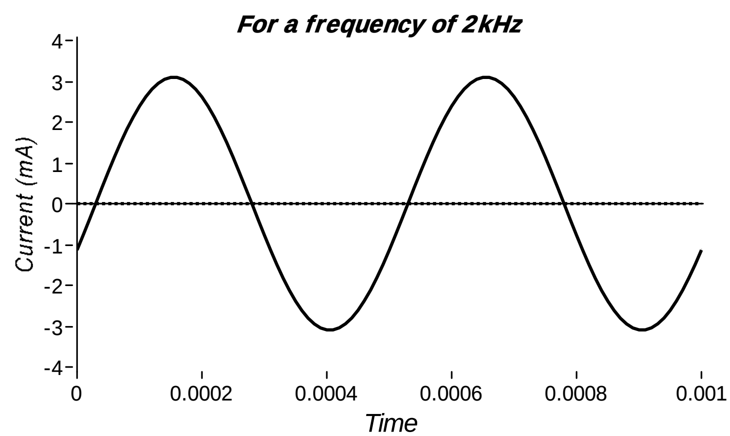
$$i(t) = \frac{3.33\times 10^{-3}}{\sqrt{1+9.8\times 10^{-10}\omega^2}} \sin\left(\omega t - \tan^{-1}\left(3.1\times 10^{-5}\omega\right)\right) \text{ Amperes}$$

Since $v_L(t) = L\frac{di(t)}{dt}$, the voltage across the inductor becomes

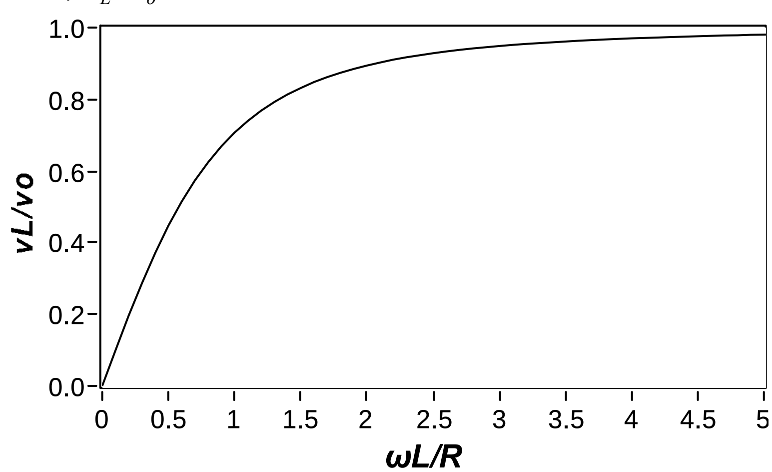
$$v_L(t) = \frac{1.57\times 10^{-4}\omega}{\sqrt{1+9.8\times 10^{-10}\omega^2}} \cos\left(\omega t - \tan^{-1}\left(3.1\times 10^{-5}\omega\right)\right) \text{ Amperes}$$

The plots for the current $i(t)$ and voltage $v_L(t)$ are shown on the figures below for a signal frequency of 2kHz. At this frequency the phase is -21.49 degrees and the amplitude of $v_L(t)$ is 1.83 Volts and the amplitude of the current $i(t)$ is 3.1 mA





The following figure shows the plot of the ratio v_L/v_o as a function of the parameter $\omega L/R$. Note that at $\omega L/R = 1$, $v_L / v_o = 1/\sqrt{2}$



The Phasor:
[Not a physical quantity: just a compact representation of a complex number]

The phasor is another way of representing the complex source and response of a system. It is a compact representation of a signal in the frequency domain in which only the magnitude and the phase are shown explicitly. The frequency is implicitly included in the representation.

Here is a summary of how a signal is represented in the time and the frequency domains.

$$\underbrace{v(t) = A\cos(\omega t + \phi) \rightarrow \quad v(t) = \mathrm{Re}\left\{Ae^{j(\omega t+\phi)}\right\} \rightarrow}_{\text{Time domain}}$$

$$\underbrace{\underline{V} = Ae^{j\phi} \rightarrow \quad \underline{V} = A\angle\phi \quad \text{(phasor)}}_{\text{Frequency domain}}$$

The parameter $\underline{V}$ is called the phasor and it may be written in the exponential form or with the angle notation as indicated above.

The usefulness of the phasor notation will become apparent as we look at how the voltage-current relationships for the inductor and the capacitor are represented in the frequency domain.

The current-voltage relationship for a capacitor in the time domain is

$$\boxed{i(t) = C\frac{dv(t)}{dt}} \tag{1.13}$$

The corresponding schematic is shown on Figure 3.

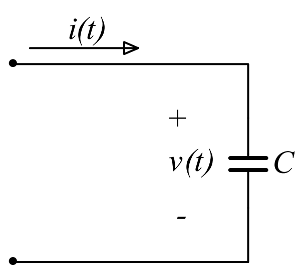


Figure 3

By applying a complex voltage of the form

$$v(t) = v_o e^{j(\omega t+\theta)} \tag{1.14}$$

The current flowing through the capacitor is

$$i(t) = i_o e^{j(\omega t+\phi)} \tag{1.15}$$

Note that signals represented by equations (1.14) and (1.15) have the same frequency but different phases.

The current-voltage relationship for the capacitor becomes

$$i_o e^{j(\omega t+\phi)} = C \frac{d\left(v_o e^{j(\omega t+\theta)}\right)}{dt} \tag{1.16}$$

which upon simplification gives

$$\underbrace{i_o e^{j\phi}}_{I} = j\omega C \underbrace{v_o e^{j\theta}}_{V} \tag{1.17}$$

Equation (1.17) may now be written in compact phasor notation as follows

$$\boxed{I = j\omega C V} \tag{1.18}$$

The corresponding circuit is shown on Figure 4.

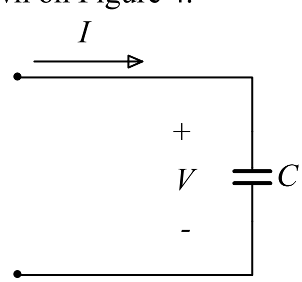


Figure 4

Equation (1.18) represents the behavior of the capacitor in the frequency domain and it is equivalent to the time domain representation given by Equation (1.13). Note that by working in the frequency domain we have reduced the differential relationship given by Equation (1.13) to an equivalent algebraic relationship, Equation (1.18).

Similarly, the time domain current-voltage relationship for an inductor may be reduced to the corresponding relationship in the frequency domain. Figure 5 shows the circuit representation in the time domain.

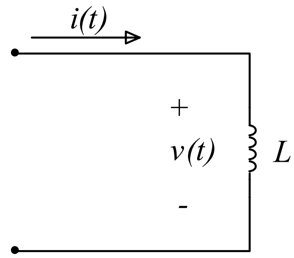


Figure 5

The current-voltage relationship for the inductor in the time domain is

$$v(t) = L\frac{di(t)}{dt} \tag{1.19}$$

By using again the complex forcing functions for the voltage and the current, Equations (1.14) and (1.15) respectively, the current-voltage relationship given by Equation (1.19) becomes

$$v_o e^{j(\omega t+\theta)} = L\frac{d\left(i_o e^{j(\omega t+\phi)}\right)}{dt} \tag{1.20}$$

which simplifies to

$$\underbrace{v_o e^{j\theta}}_{V} = j\omega L \underbrace{i_o e^{j\phi}}_{I} \tag{1.21}$$

$$\boxed{V = j\omega L\, I} \tag{1.22}$$

Again we have reduced the differential relationship given by Equation (1.19) to an equivalent algebraic relationship, Equation (1.22). The corresponding circuit is shown on Figure 6

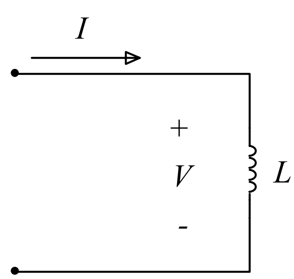
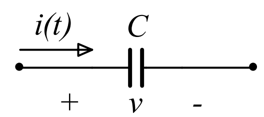
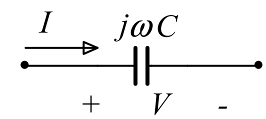
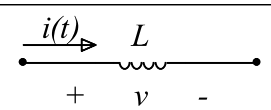
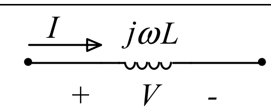
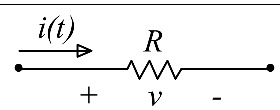
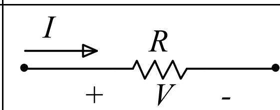


Figure 6

Equivalently we may also show that for the resistor, the frequency domain representation has the same form as the time domain representation.

The table below summarizes the characteristic relationships between voltage and current for the capacitor, the inductor and the resistor in the time and the frequency domain.

Time domain		Frequency domain	
Relationship	Symbol	Relationship	Symbol
$i(t) = C\dfrac{dv(t)}{dt}$		$V = \dfrac{1}{j\omega C} I$	
$v(t) = L\dfrac{di(t)}{dt}$		$V = j\omega LI$	
$v(t) = Ri(t)$		$V = RI$	

Impedance

The ratios of V/I in the frequency domain for the resistor, the capacitor and the inductor are:

$$
\begin{aligned}
&\text{Resistor:} && \frac{V}{I} = R \\
&\text{Capacitor:} && \frac{V}{I} = \frac{1}{j\omega C} \\
&\text{Inductor:} && \frac{V}{I} = j\omega L
\end{aligned}
\tag{1.23}
$$

These ratios are called **impedance** and it is most often given the symbol Z.

$$
\begin{aligned}
&\text{Impedance of a Resistor:} && Z_R = R \\
&\text{Impedance of a Capacitor:} && Z_C = \frac{1}{j\omega C} \\
&\text{Impedance of an Inductor:} && Z_L = j\omega L
\end{aligned}
\tag{1.24}
$$

Impedance is a frequency dependent quantity, it has the units of Ω and it is a complex number. It has meaning only in the frequency domain and cannot be transformed directly to the time domain. Indeed the impedance of a device represents the opposition(resistance) to the flow of sinusoidal current through the device. In general any device may be described in terms of its impedance *Z* like:

$$V = Z\,I \qquad (1.25)$$

and schematically as shown on Figure 7.

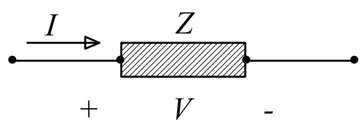


Figure 7

Impedance is a very powerful concept and it enables us to drastically simplify and analyze circuits.

Impedances combine in the same way that resistors do:

Impedances in series add to an equivalent impedance Z_{eq} like: $Z_{eq} = Z_1 + Z_2 + \ldots Z_n$

Impedances in parallel add to an equivalent impedance Z_{eq} like: $\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \ldots \frac{1}{Z_n}$

Methods of Analysis:

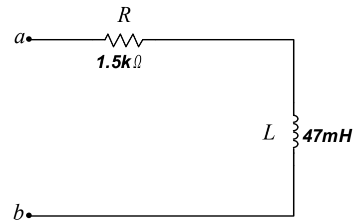
Since we still dealing with linear circuits we may employ:

Superposition
Thevenin and Norton theorems
Source transformations

The resistance is replaced by the impedance of the elements

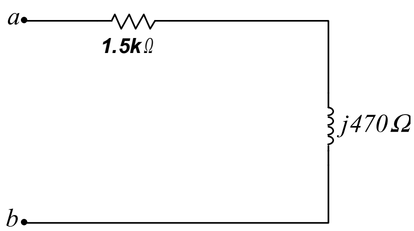
Example:

Calculate the equivalent impedance of the following circuit viewed through port a-b assuming that the circuit operates with a frequency ω of 10000rad/sec.



The impedance of the resistor is 1.5kΩ and the impedance of the inductor is
$j\omega L = j(10000)47\times10^{-3}\Omega = j\,470\Omega$.

In terms of the impedance the circuit is now given below.
The equivalent impedance is now given by the summation of the two impedances across terminals a and b.



$$Z_{eq} = (1500 + j470)\ \Omega$$

Example:
Voltage $v = 5\sin(10000t - \pi/3)$ is applied across a capacitor of capacitance 47μF. What is the expression for the current flowing through the capacitor?

Using the concept of impedance for a capacitor we have:

$$V = \frac{1}{j\omega C} I = \frac{1}{j0.47} I$$

The voltage (in phasor form) is $V = 5\angle -60^0$.
The complex number j0.47 may also be written in phasor form as $0.47\angle 90^0$

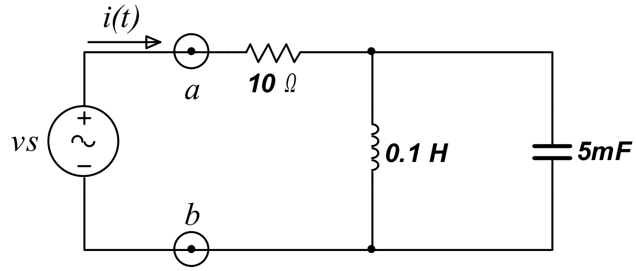
And so the current is (frequency domain)

$$I = 5(0.47)\angle 30^0$$

And so the current flowing through the capacitor (time domain)

$$i(t) = 2.35\sin(10000t + \pi/3)$$

Example:
Calculate the current *i(t)* as indicated in the circuit below for $vs = 5\sin(10t - 20^0)$



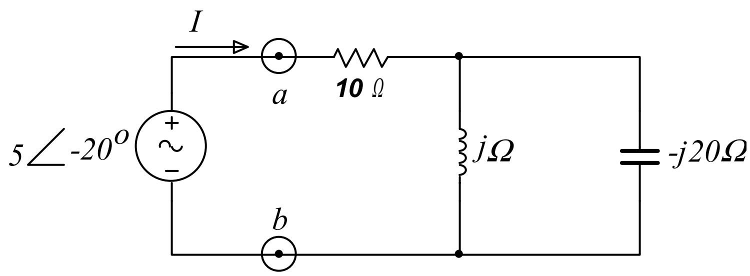
First we identify the frequency of the system: ω=10 rad/sec
Next we calculate the impedance of each component.

0.1H inductor: $Z_L = j10(0.1) = j\ \Omega$

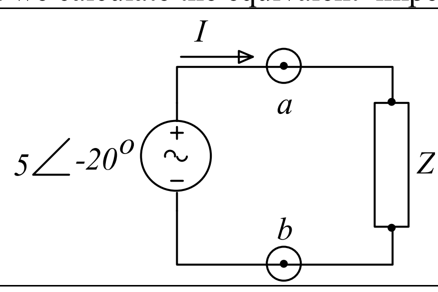
5mF capacitor: $Z_C = \dfrac{-j}{10(0.005)} = -j\,20\ \Omega$

10Ω resistor: $Z_R = 10\ \Omega$

In the frequency domain the circuit becomes:



Next we calculate the equivalent impedance Z seen across the terminals a-b



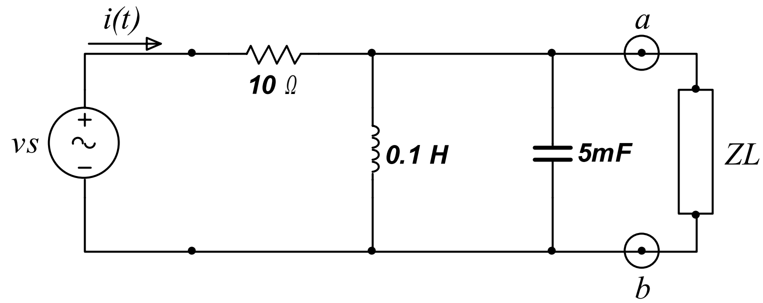
$$\begin{aligned} Z &= 10\Omega + (j\Omega)//(-j20\Omega) \\ &= 10 + j\frac{20}{19}\ \Omega \\ &= 10.055e^{j6^0}\ \Omega \\ &= 10.055\angle 6^0\ \Omega \end{aligned}$$

And so the current I becomes: $I = \dfrac{V}{Z} = \dfrac{5\angle -20^0}{10.055\angle 6^0} = 0.497\angle -24^0$

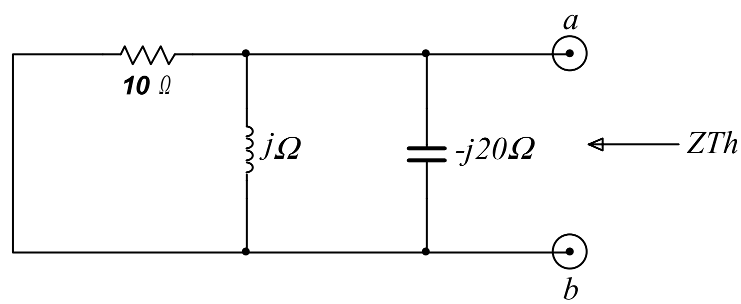
And so $i(t) = 0.497\sin(10t - 24^0)$ *Amperes*

Example:

Determine the Thevenin equivalent circuit seen by the load *ZL*. Assume that $vs = 5\sin(10t - 20^0)$



The Thevenin resistance is calculated by zeroing out the independent source as shown below



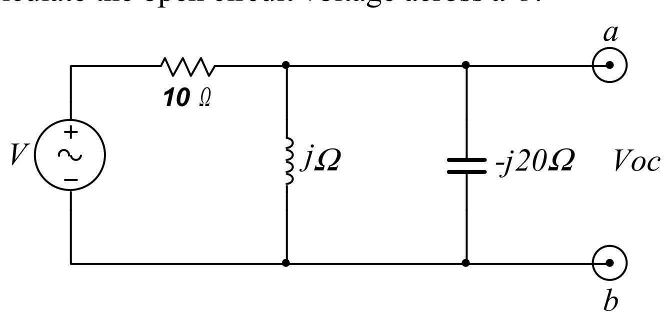
And *ZTh* is given by

$$\frac{1}{ZTh} = \frac{1}{10} + \frac{1}{j} + \frac{1}{-j20}$$

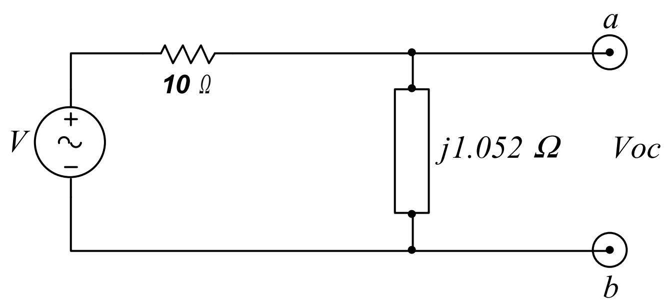
$$ZTh = \frac{10j(-j20)}{10 + j(1-20)} = \frac{200}{10 - j19} = \frac{200\angle 0^0}{21.47\angle -62.2^0} = 9.31\angle 62.2^0\ \Omega$$

$$= 4.43 + j8.23\ \ \Omega$$

Now we need to calculate the open circuit voltage across *a-b*.



By combining the *j* and the *–j20* impedances we have the following circuit



And from the voltage divider rule:

$$
\begin{aligned}
Voc &= V\frac{j1.052\,\Omega}{10+j1.052\,\Omega} \\
&= V\left(0.104\angle -6^0\right)\,Volts \\
&= 5\angle -20^0\;0.104\angle -6^0\;\;Volts \\
&= 0.52\angle -26^0
\end{aligned}
$$

And thus the Thevenin equivalent circuit is