Filter design

- FIR filters
- Chebychev design
- linear phase filter design
- equalizer design
- filter magnitude specifications

FIR filters

finite impulse response (FIR) filter:

$$y(t) = \sum_{\tau=0}^{n-1} h_\tau u(t-\tau), \quad t \in \mathbf{Z}$$

- (sequence) $u : \mathbf{Z} \to \mathbf{R}$ is *input signal*
- (sequence) $y : \mathbf{Z} \to \mathbf{R}$ is *output signal*
- h_i are called *filter coefficients*
- n is filter *order* or *length*

filter frequency response: $H : \mathbf{R} \to \mathbf{C}$

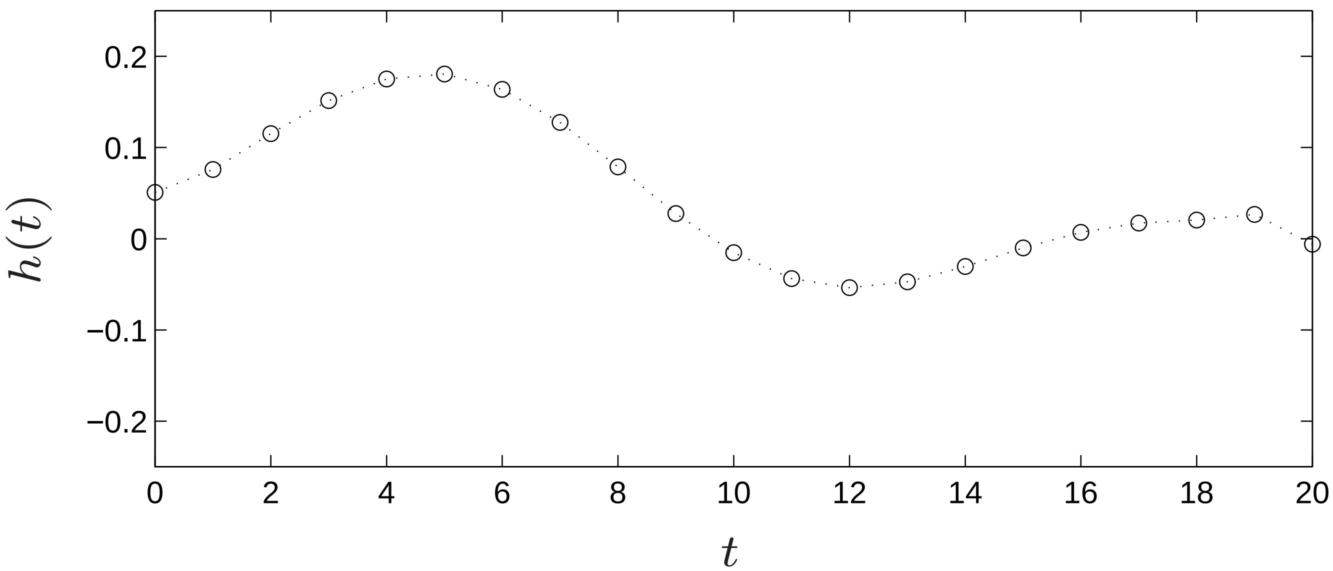
$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-i\omega} + \cdots + h_{n-1} e^{-i(n-1)\omega} \\ &= \sum_{t=0}^{n-1} h_t \cos t\omega + i \sum_{t=0}^{n-1} h_t \sin t\omega \end{aligned}$$

- (EE tradition uses $j = \sqrt{-1}$ instead of i)
- H is periodic and conjugate symmetric, so only need to know/specify for $0 \le \omega \le \pi$

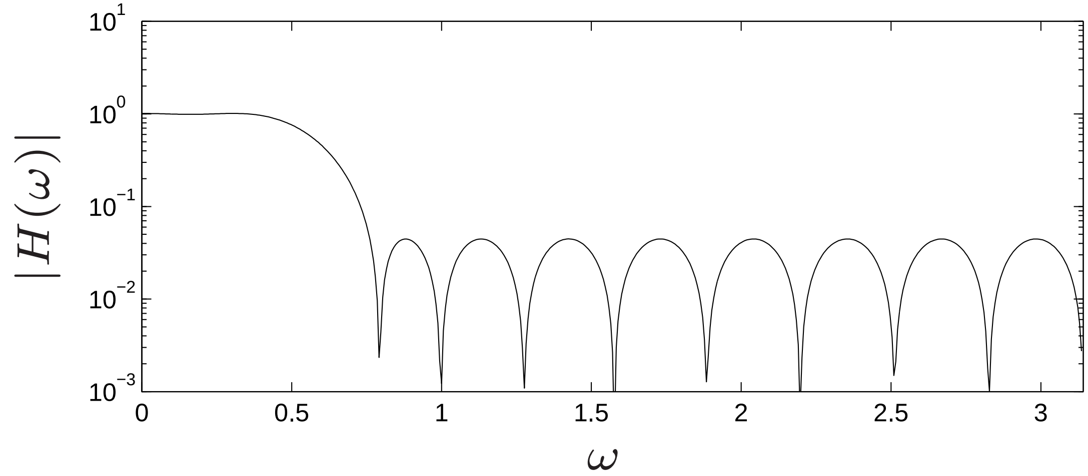
FIR filter design problem: choose h so it and H satisfy/optimize specs

example: (lowpass) FIR filter, order $n = 21$

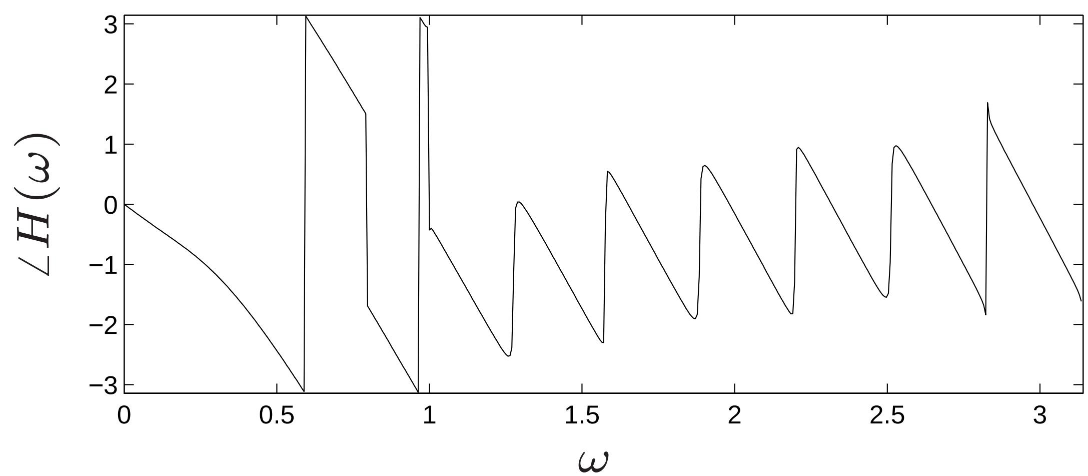
impulse response h:



frequency response magnitude (*i.e.*, $|H(\omega)|$):



frequency response phase (*i.e.*, $\angle H(\omega)$):



Chebychev design

$$\text{minimize } \max_{\omega\in[0,\pi]} |H(\omega) - H_{\text{des}}(\omega)|$$

- h is optimization variable
- $H_{\text{des}} : \mathbf{R} \to \mathbf{C}$ is (given) **desired transfer function**
- convex problem
- can add constraints, *e.g.*, $|h_i| \leq 1$

sample (discretize) frequency:

$$\text{minimize } \max_{k=1,\ldots,m} |H(\omega_k) - H_{\text{des}}(\omega_k)|$$

- sample points $0 \leq \omega_1 < \cdots < \omega_m \leq \pi$ are fixed ($e.g.$, $\omega_k = k\pi/m$)
- $m \gg n$ (common rule-of-thumb: $m = 15n$)
- yields approximation (relaxation) of problem above

Chebychev design via SOCP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left\| A^{(k)}h - b^{(k)} \right\| \le t, \quad k = 1, \ldots, m \end{array}$$

where

$$\begin{aligned} A^{(k)} &= \begin{bmatrix} 1 & \cos\omega_k & \cdots & \cos(n-1)\omega_k \\ 0 & -\sin\omega_k & \cdots & -\sin(n-1)\omega_k \end{bmatrix} \\ b^{(k)} &= \begin{bmatrix} \Re H_{\rm des}(\omega_k) \\ \Im H_{\rm des}(\omega_k) \end{bmatrix} \\ h &= \begin{bmatrix} h_0 \\ \vdots \\ h_{n-1} \end{bmatrix} \end{aligned}$$

Linear phase filters

suppose

- $n = 2N + 1$ is odd
- impulse response is symmetric about midpoint:

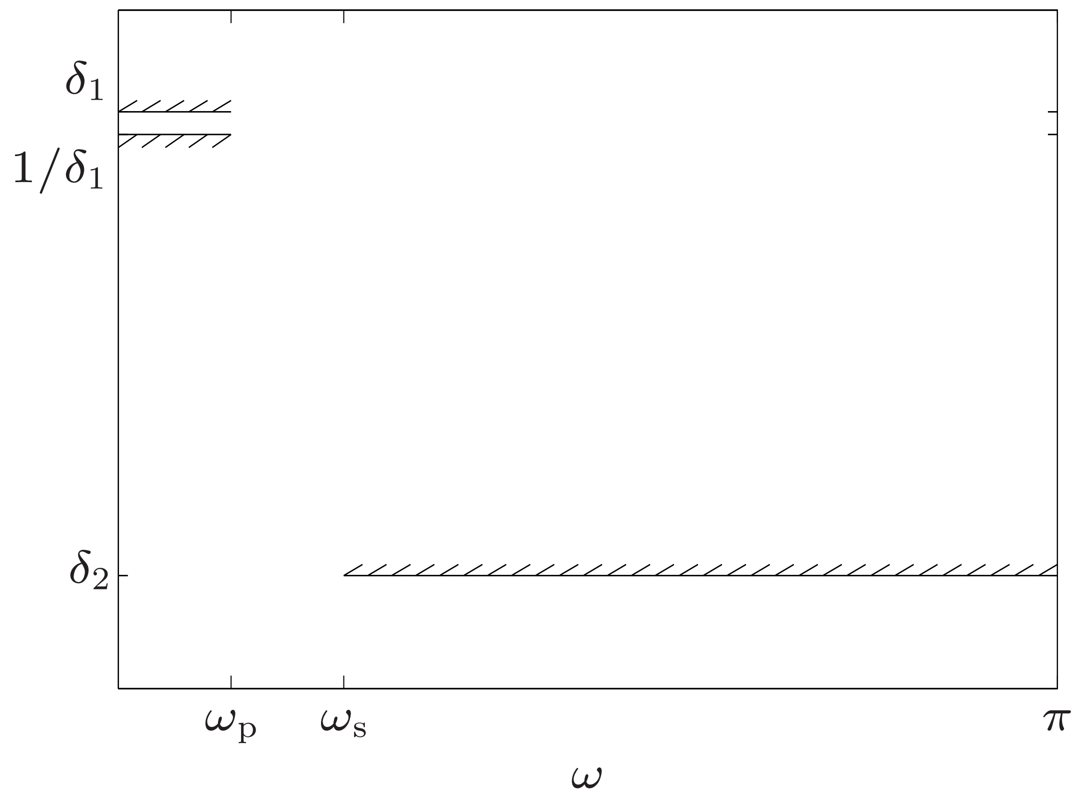
$$h_t = h_{n-1-t}, \quad t = 0, \ldots, n-1$$

then

$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-i\omega} + \cdots + h_{n-1} e^{-i(n-1)\omega} \\ &= e^{-iN\omega} \left(2h_0 \cos N\omega + 2h_1 \cos(N-1)\omega + \cdots + h_N \right) \\ &\stackrel{\Delta}{=} e^{-iN\omega} \widetilde{H}(\omega) \end{aligned}$$

- term $e^{-iN\omega}$ represents N-sample delay
- $\widetilde{H}(\omega)$ is **real**
- $|H(\omega)| = |\widetilde{H}(\omega)|$
- called **linear phase** filter ($\angle H(\omega)$ is linear except for jumps of $\pm\pi$)

Lowpass filter specifications



idea:

- pass frequencies in *passband* $[0, \omega_{\mathrm{p}}]$
- block frequencies in *stopband* $[\omega_{\mathrm{s}}, \pi]$

specifications:

- maximum *passband ripple* ($\pm 20 \log_{10} \delta_1$ in dB):

$$1/\delta_1 \leq |H(\omega)| \leq \delta_1, \quad 0 \leq \omega \leq \omega_{\mathrm{p}}$$

- minimum *stopband attenuation* ($-20 \log_{10} \delta_2$ in dB):

$$|H(\omega)| \leq \delta_2, \quad \omega_{\mathrm{s}} \leq \omega \leq \pi$$

Linear phase lowpass filter design

- sample frequency
- can assume wlog $\widetilde{H}(0) > 0$, so ripple spec is

$$1/\delta_1 \leq \widetilde{H}(\omega_k) \leq \delta_1$$

design for **maximum stopband attenuation:**

$$\begin{array}{ll} \text{minimize} & \delta_2 \\ \text{subject to} & 1/\delta_1 \leq \widetilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_\mathrm{p} \\ & -\delta_2 \leq \widetilde{H}(\omega_k) \leq \delta_2, \quad \omega_\mathrm{s} \leq \omega_k \leq \pi \end{array}$$

- passband ripple δ_1 is given
- an LP in variables h, δ_2
- known (and used) since 1960's
- can add other constraints, $e.g.$, $|h_i| \leq \alpha$

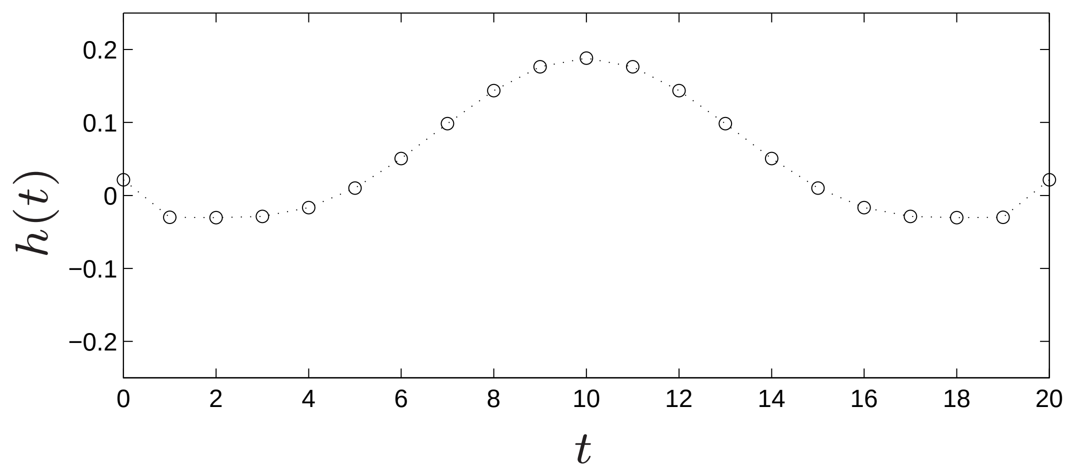
variations and extensions:

- fix δ_2, minimize δ_1 (convex, but not LP)
- fix δ_1 and δ_2, minimize ω_{s} (quasiconvex)
- fix δ_1 and δ_2, minimize order n (quasiconvex)

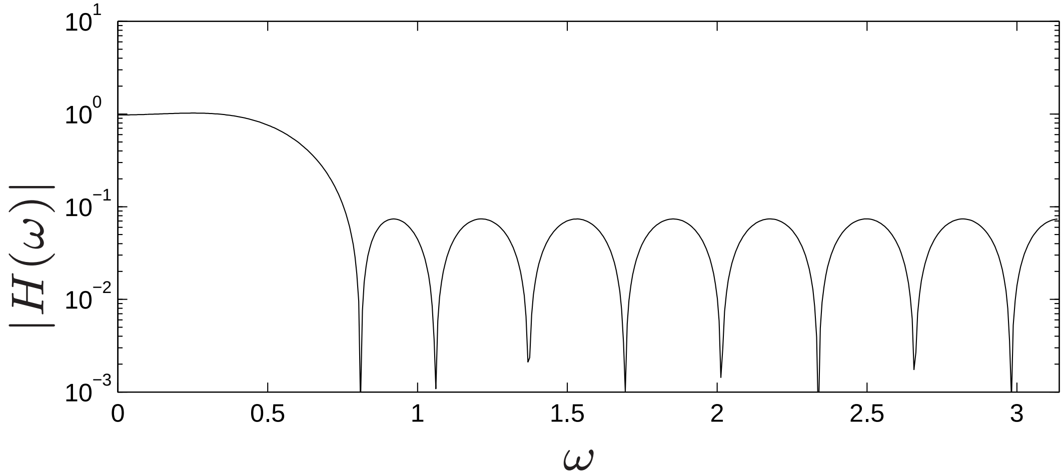
example

- linear phase filter, $n = 21$
- passband $[0, 0.12\pi]$; stopband $[0.24\pi, \pi]$
- max ripple $\delta_1 = 1.012$ (± 0.1dB)
- design for maximum stopband attenuation

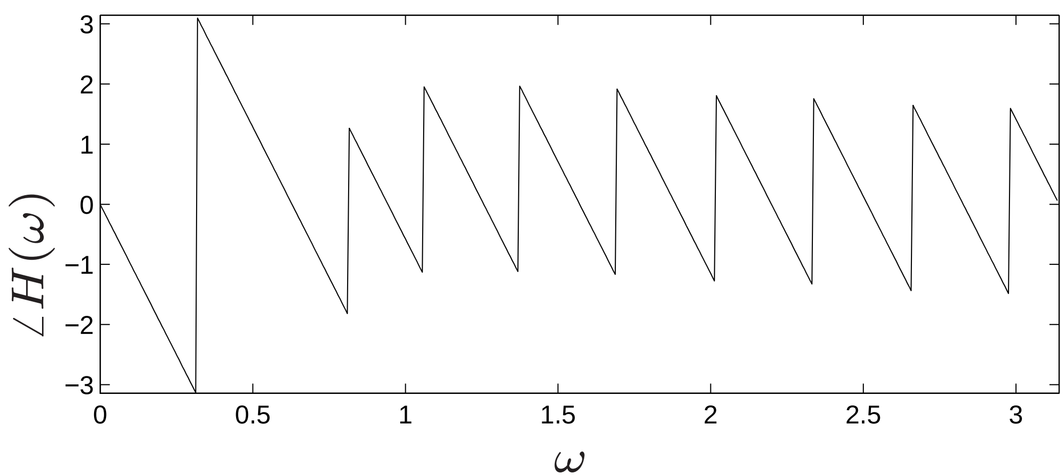
impulse response h:



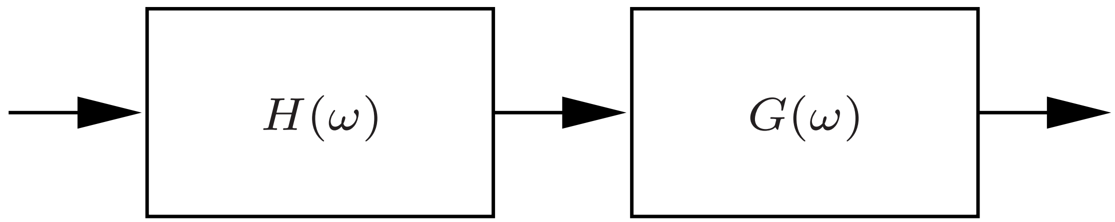
frequency response magnitude ($i.e.$, $|H(\omega)|$):



frequency response phase ($i.e.$, $\angle H(\omega)$):



Equalizer design



equalization: given

- G (unequalized frequency response)
- G_{des} (desired frequency response)

design (FIR equalizer) H so that $\widetilde{G} \stackrel{\Delta}{=} GH \approx G_{\text{des}}$

- common choice: $G_{\text{des}}(\omega) = e^{-iD\omega}$ (delay)
  $i.e.$, equalization is deconvolution (up to delay)
- can add constraints on H, $e.g.$, limits on $|h_i|$ or $\max_\omega |H(\omega)|$

Chebychev equalizer design:

$$\text{minimize} \max_{\omega\in[0,\pi]} \left|\widetilde{G}(\omega) - G_{\text{des}}(\omega)\right|$$

convex; SOCP after sampling frequency

time-domain equalization: optimize impulse response $\tilde{g}$ of equalized system
e.g., with $G_{\rm des}(\omega) = e^{-iD\omega}$,

$$g_{\rm des}(t) = \begin{cases} 1 & t = D \\ 0 & t \neq D \end{cases}$$

sample design:

$$\begin{array}{ll} \text{minimize} & \max_{t \neq D} |\tilde{g}(t)| \\ \text{subject to} & \tilde{g}(D) = 1 \end{array}$$

- an LP
- can use $\sum_{t \neq D} \tilde{g}(t)^2$ or $\sum_{t \neq D} |\tilde{g}(t)|$

extensions:

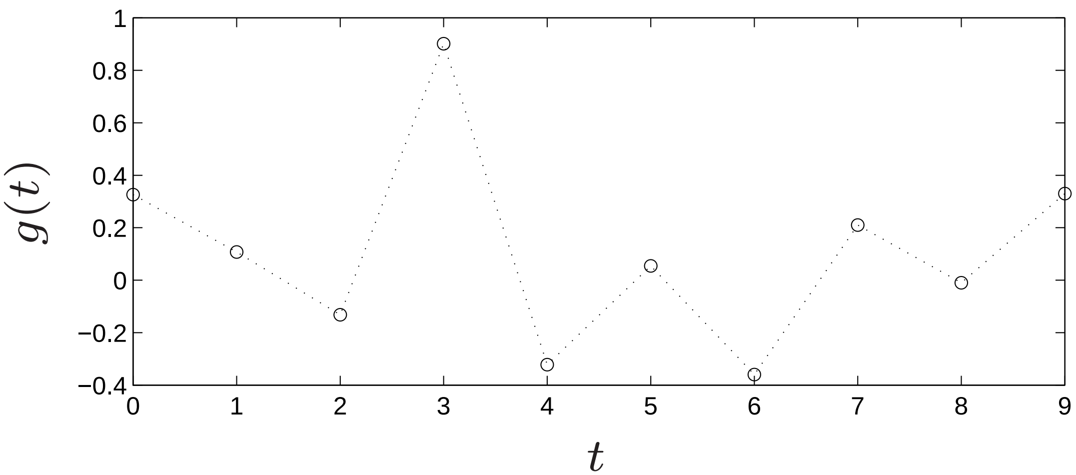
- can impose (convex) constraints
- can mix time- and frequency-domain specifications
- can equalize multiple systems, $i.e.$, choose H so

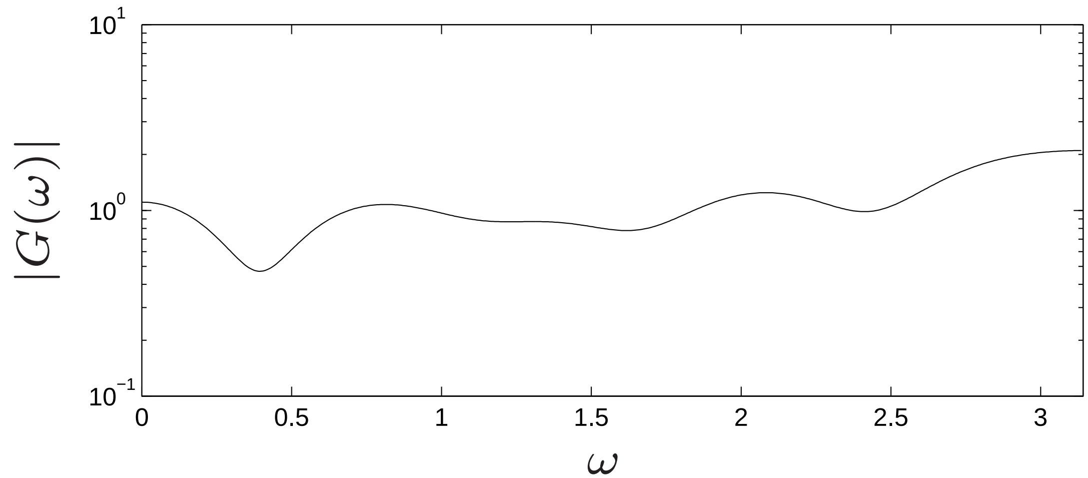
$$G^{(k)}H \approx G_{\mathrm{des}}, \quad k = 1, \ldots, K$$

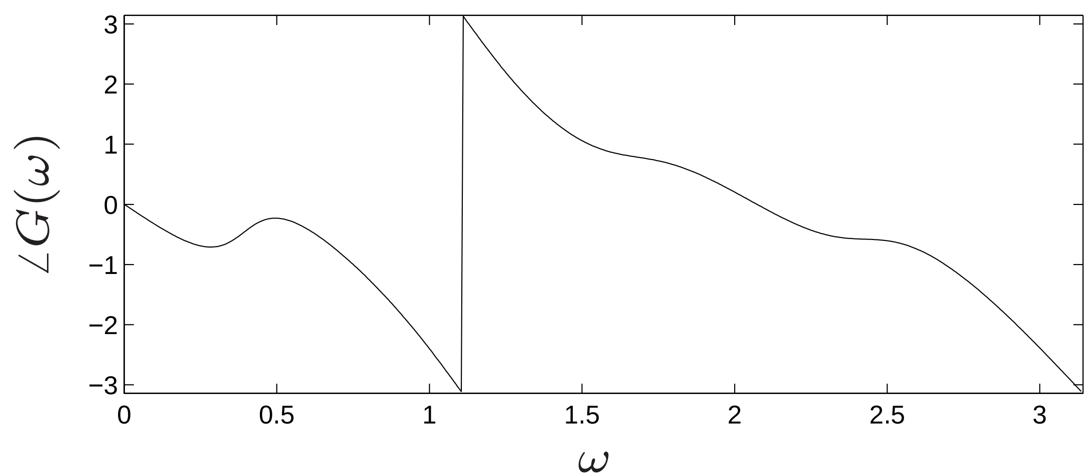
- can equalize multi-input multi-output systems ($i.e.$, G and H are matrices)
- extends to multidimensional systems, $e.g.$, image processing

Equalizer design example

unequalized system G is 10th order FIR:





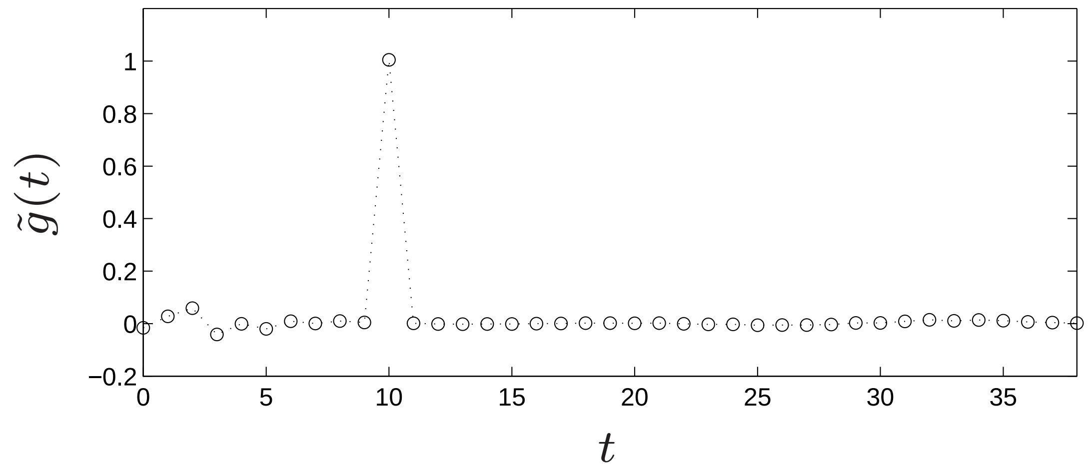


design 30th order FIR equalizer with $\widetilde{G}(\omega) \approx e^{-i10\omega}$

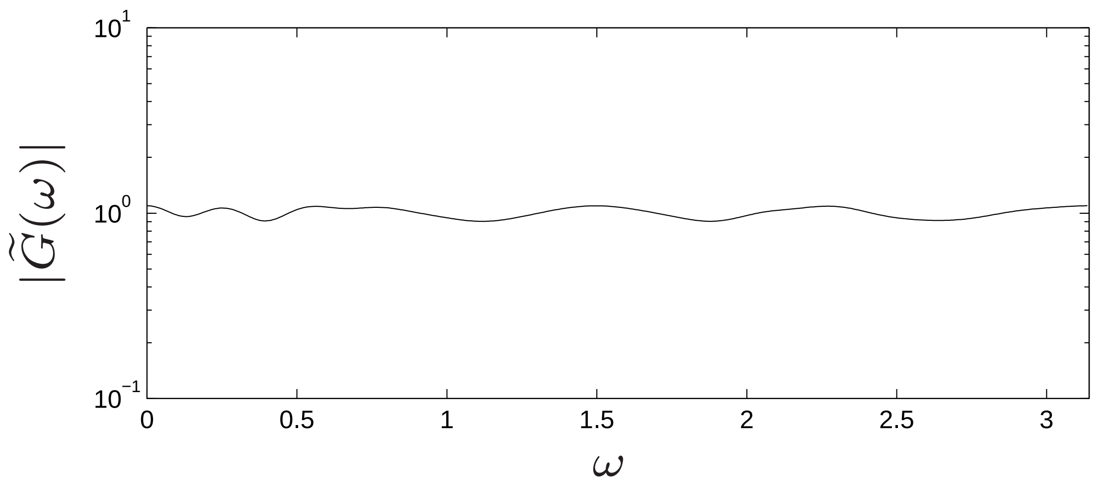
Chebychev equalizer design:

$$\text{minimize} \;\; \max_{\omega} \left| \tilde{G}(\omega) - e^{-i10\omega} \right|$$

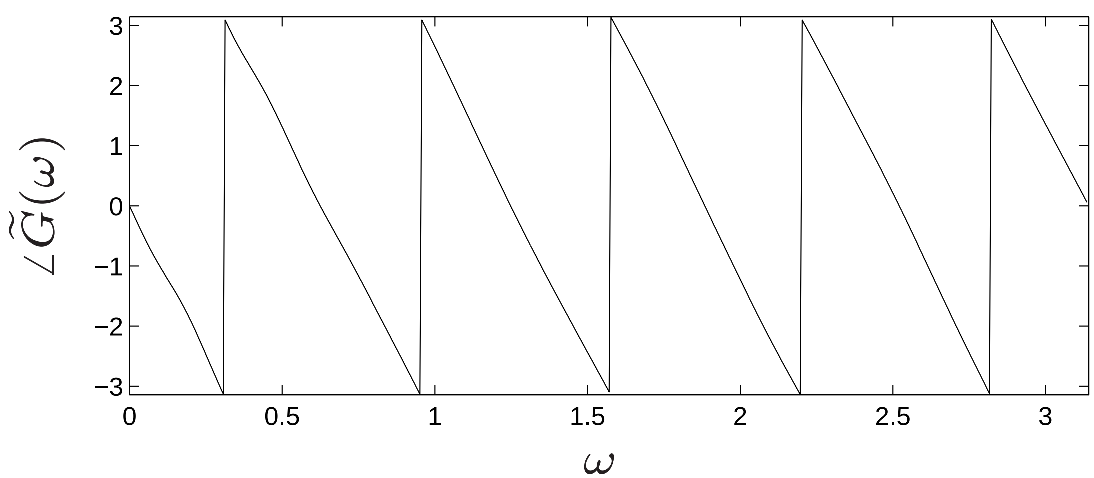
equalized system impulse response $\tilde{g}$



equalized frequency response magnitude $|\widetilde{G}|$



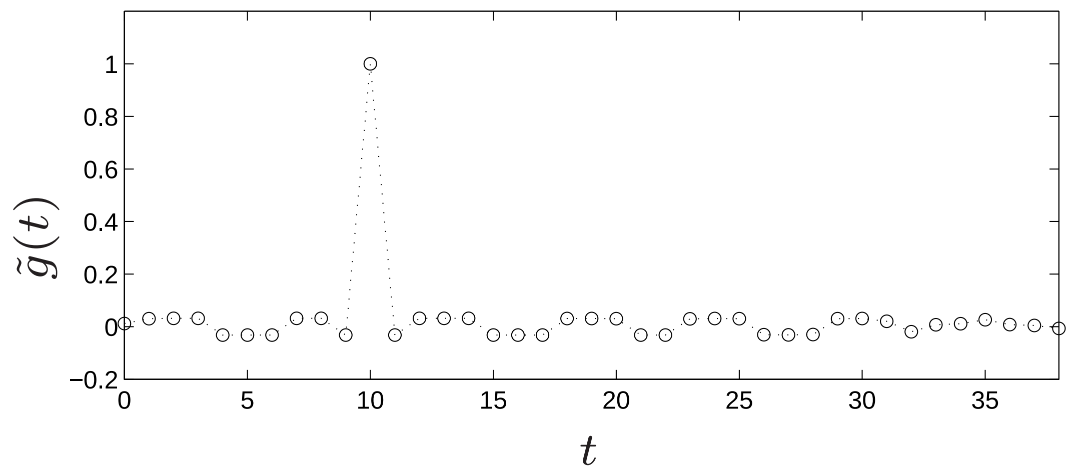
equalized frequency response phase $\angle\widetilde{G}$



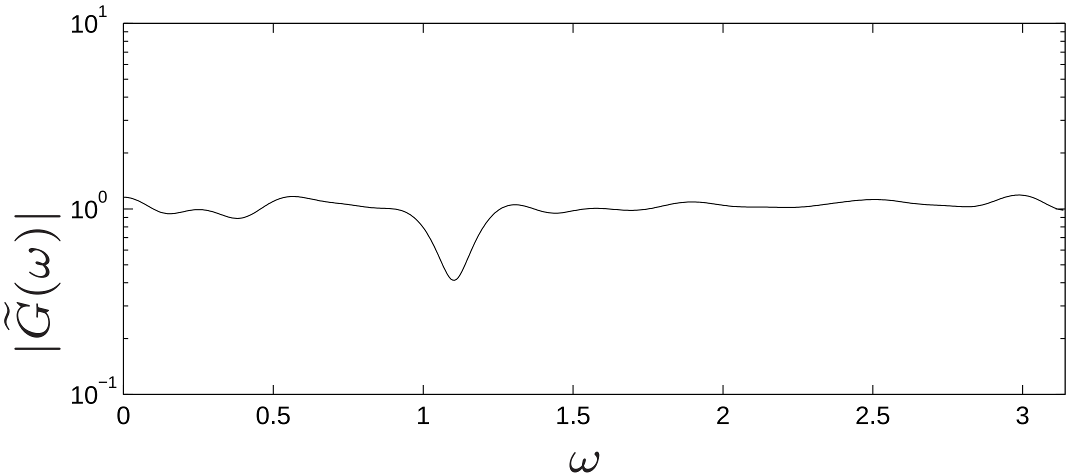
time-domain equalizer design:

$$\text{minimize} \quad \max_{t \neq 10} |\tilde{g}(t)|$$

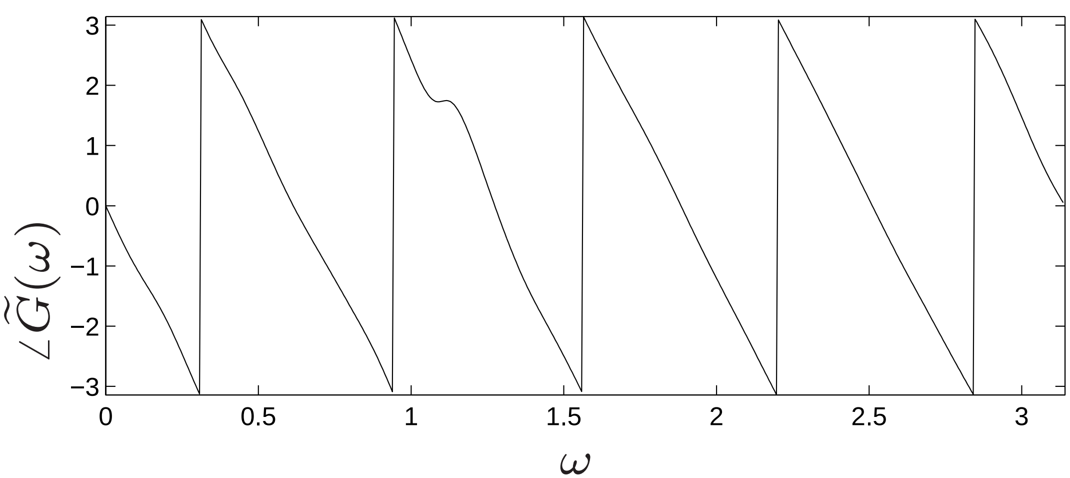
equalized system impulse response $\tilde{g}$



equalized frequency response magnitude $|\widetilde{G}|$



equalized frequency response phase $\angle\widetilde{G}$



Filter magnitude specifications

transfer function *magnitude spec* has form

$$L(\omega) \leq |H(\omega)| \leq U(\omega), \quad \omega \in [0, \pi]$$

where $L, U : \mathbf{R} \to \mathbf{R}_+$ are given

- lower bound is **not** convex in filter coefficients h
- arises in many applications, $e.g.$, audio, spectrum shaping
- can change variables to solve via convex optimization

Autocorrelation coefficients

autocorrelation coefficients associated with impulse response $h = (h_0, \ldots, h_{n-1}) \in \mathbf{R}^n$ are

$$r_t = \sum_\tau h_\tau h_{\tau+t}$$

(we take $h_k = 0$ for $k < 0$ or $k \geq n$)

- $r_t = r_{-t}$; $r_t = 0$ for $|t| \geq n$
- hence suffices to specify $r = (r_0, \ldots, r_{n-1}) \in \mathbf{R}^n$

Fourier transform of autocorrelation coefficients is

$$R(\omega) = \sum_{\tau} e^{-i\omega\tau} r_\tau = r_0 + \sum_{t=1}^{n-1} 2 r_t \cos \omega t = |H(\omega)|^2$$

- always have $R(\omega) \geq 0$ for all ω
- can express magnitude specification as

$$L(\omega)^2 \leq R(\omega) \leq U(\omega)^2, \quad \omega \in [0, \pi]$$

. . . **convex** in r

Spectral factorization

question: when is $r \in \mathbf{R}^n$ the autocorrelation coefficients of some $h \in \mathbf{R}^n$?

answer: (*spectral factorization theorem*) if and only if $R(\omega) \geq 0$ for all ω

- spectral factorization condition is convex in r
- many algorithms for spectral factorization, $i.e.$, finding an h s.t. $R(\omega) = |H(\omega)|^2$

magnitude design via autocorrelation coefficients:

- use r as variable (instead of h)
- add spectral factorization condition $R(\omega) \geq 0$ for all ω
- optimize over r
- use spectral factorization to recover h

log-Chebychev magnitude design

choose h to minimize

$$\max_{\omega} \; |20\log_{10}|H(\omega)| - 20\log_{10} D(\omega)|$$

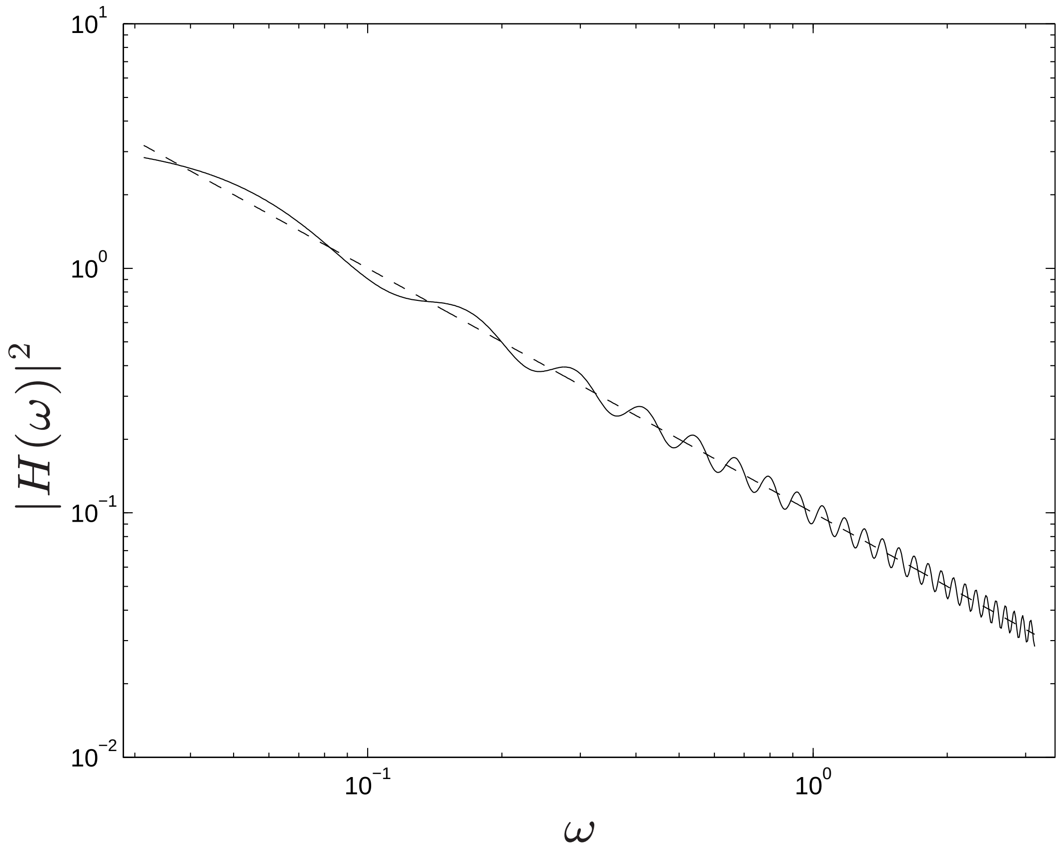
- D is desired transfer function magnitude ($D(\omega) > 0$ for all ω)
- find minimax logarithmic (dB) fit

reformulate as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & D(\omega)^2/t \leq R(\omega) \leq tD(\omega)^2, \quad 0 \leq \omega \leq \pi \end{array}$$

- convex in variables r, t
- constraint includes spectral factorization condition

example: $1/f$ (pink noise) filter ($i.e.$, $D(\omega) = 1/\sqrt{\omega}$), $n = 50$, log-Chebychev design over $0.01\pi \leq \omega \leq \pi$



optimal fit: ± 0.5dB