MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Spring 2011

HOMEWORK 7 SOLUTIONS

Exercise 15.1 (a) The system is causal if the impulse response is right-sided. Consider a sequence $e^{-at}u[t]$, where $u[t]$ is a unit step: $u[t] = 1$ for $t \geq 0$, and zero otherwise. Laplace tranform of this sequence converges if $Re(s) > -a$, and is equal to

$$\int_{-\infty}^{\infty} e^{-st}e^{-at}u[t]dt = \frac{1}{s+a}, \text{ ROC} : Re(s) > -a$$

Therefore for a system represented by first-order transfer function to be causal the ROC has to be to the right of the pole (in fact this is true for a multiple pole as well). Since a rational function can be represented by a partial fraction expansion, and region of convergence is defined by the intersection of individual regions of convergence, the ROC of the system has to lie to the right of the rightmost pole for the system to be causal. In case of a rational transfer function this is also a sufficient condition. Note that if an LTI system has a rational transfer function, its impulse response consists of diverging or decaying exponents (maybe multiplied by powers of t), therefore all concepts of p-stability are equivalent. For BIBO stability the impulse response has to be absolutely integrable, which is equivalent to existence of Fourier transform. The Fourier transform is Laplace transform evaluated at the $j\omega$ axis. Therefore for stability the ROC has to include the $j\omega$ axis. Using these two rules we can see that the system

$$G(s) = \frac{s+2}{(s-2)(s+1)}$$

- (i) is neither causal nor stable for ROC given by $Re(s) < -1$
- (ii) is non-causal and stable for ROC $-1 < Re(s) < 2$.
- (iii) is causal and unstable for ROC $Re(s) > 2$

Another way to solve the problem is to find (look up in the tables) inverse Laplace tranforms corresponding to the transfer function and ROC pairs. Compute partial fraction expansion

$$G(s) = \frac{s+2}{(s-2)(s+1)} = \frac{1}{3}\left[\frac{4}{s-2} - \frac{1}{s+1}\right]$$

The impulse response functions in three different cases are

$$\begin{aligned}
\text{(i) } h(t) &= \frac{1}{3}\left[-4e^{2t}u[-t] + e^{-t}u[-t]\right], Re(s) < -1 \text{ anticausal, unstable} \\
\text{(ii) } h(t) &= \frac{1}{3}\left[-4e^{2t}u[-t] + e^{-t}u[t]\right], -1 < Re(s) < 2 \text{ non-causal, stable} \\
\text{(iii) } h(t) &= \frac{1}{3}\left[4e^{2t}u[t] + e^{-t}u[t]\right], Re(s) > 2 \text{ causal, unstable}
\end{aligned}$$

(b) Note that if there is a diverging exponent in the impulse response, an input which is non-zero on some interval will result in an exponentially diverging output. For example, in case (iii) choose

$f(t) = 1$ for $0 < t < 1$ and 0 otherwise. The output for any positive t will be a linear combination of e^{2t} and e^{-t}. For example for $t > 1$:

$$y(t) = \frac{1}{3}\int_{-\infty}^{+\infty} \left(4e^{2(t-\tau)} - e^{-(t-\tau)}\right) u[t-\tau] f(\tau) d\tau = \frac{2}{3}e^{2t}\left(1 - e^{-2}\right) - \frac{1}{3}e^{-t}(e-1)$$

Clearly this function grows unbounded and has an infinite p-norm. However the input $f(t)$ has p-norm equal to 1 for any p including ∞. In case (i) we can use $f(t) = 1$ for $-1 < t < 0$, for example. Infinitely long bounded inputs that do not cancel an unstable pole will also result in unbounded output. For example, choose $e^{-2t}, t \geq 0$ in case (iii).

Exercise 15.2 a) When $g(x) = cos(x)$, the system is unstable for $p \geq 1$. Proof: Suppose the system is p-stable. Then, there there exists a constant C such that $\|g\|_p \leq C\|u\|_p$. Now, with $\|z\|_p \to 0$, there exists a T such that $cos(z(t)) \to 1$ for all $t \geq T$. So, we have a contradiction where $\|u\|_p \to 0$, then $\|y\|_p \to 1$, which implies that there are no such a constant$C$ to satisfy the condition. Therefore the system is p-stable not all $p \geq 1$.

b) When $g(x) = sin(x)$, the system is p-stable for $p \geq 1$. Proof: Consider the Taylor series expansion of $y(t) = sin(z(t))$ about the origin. Then, we have

$$y(t) = sin(z(t)) = z(t) - \frac{1}{3}z^3(t) + H.O.T.$$

This implies that

$$\|y\|_p \leq \|z\|_p + O(\|z\|_p). \tag{1}$$

Now, because of the stability of the system from u to z , we have

$$\|z\|_p \leq C\|u\|_p \tag{2}$$

for some constant C. Thus combining Eqn (1) and (2), we have

$$\|y\|_p \leq C\|u\|_p + O(C\|u\|_p).$$

So, for all $\epsilon > 0$ there exists δ such that $O(C\|u\|_p) \leq \epsilon\|u\|_p$, which implies that $\|y\|_p \leq (C+\epsilon)\|u\|_p$. That concludes the p-stability, with $\|u\|_p < \delta$.

c) When $g(x)$ is a saturation function with a scale of 1, the system is p-stable for $p \geq 1$. Proof: Again since the system from u to z is p-stable, there exists a constant C such that $\|z\|_p \leq C\|u\|_p$. So, for all u with $\|u\|_p \leq \delta$, if we take C to be $\frac{1}{\delta}$, then we have:

$$\|z\|_P \leq C\|u\|_p \leq 1.$$

Since,

$$|g(z)| = \begin{cases} z & |z| \leq 1 \\ 1 & |z| \geq 1 \end{cases},$$

for $|z| \leq 1$ we have

$$\|y\|_p = \|z\|_P \leq C\|u\|_p \leq 1.$$

Therefore this system is p-stable for all $p \geq 1$ in $|z| < 1$.

Exercise 16.1 a) Since $u \in \mathbf{X}$, we can express u as

$$u = \sum_{i=1}^{N} u_i e^{j\omega_i t} \quad \text{where } u_i \in \mathbb{R}^n, \quad \omega_i \in \mathbb{R}.$$

With

$$\begin{aligned} u'(t) &= \sum_{i=1}^{N} e^{-j\omega_i t} = \sum_{i=1}^{N} u_i^T e^{-j\omega_i t} \\ \rightarrow u'(t)u(t) &= (\sum_{i=1}^{N} e^{-j\omega_i t} u_i^T)(\sum_{k=1}^{N} u_j e^{j\omega_k t}) \\ &= \sum_{i=1}^{N}\sum_{k=1}^{N} e^{j(\omega_k - \omega_i)t} u_i^T u_k, \end{aligned}$$

we can compute P_u as follows:

$$\begin{aligned} P_u &= \lim_{L\to\infty} \left(\frac{1}{2L} \int_{-L}^{L} u'(t)u(t)dt \right) \\ &= \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \sum_i \sum_j u_i^T u_j e^{j(\omega_j - \omega_i)t} dt \\ &= \lim_{L\to\infty} \frac{1}{2L} \sum_i \sum_j u_i^T u_j \int_{-L}^{L} e^{j(\omega_j - \omega_i)t} dt. \end{aligned}$$

Note that as $L \to \infty$, because of the orthnormality of complex exponential,

$$\lim_{L\to\infty} \int_{-L}^{L} e^{j(\omega_j - \omega_i)tdt} = \begin{cases} 0 & : \quad i \neq j \\ 1 & : \quad i = j \end{cases}$$

Thus

$$P_u = \lim_{L\to\infty} \frac{1}{2L} \sum_{i=1}^{N} u_i^T u_i (2L) = \sum_{i=1}^{N} \|u_i\|_2^2.$$

b) The output of the system can be expressed as $\underline{y}(t) = \mathcal{H}(t) * \underline{u}(t)$ in time domain or $Y(s) = H(s)U(s)$ in frequency domain. For a CT LTI system, we have $y = H(j\omega_i)u_i e^{j\omega_i t}$ if $u = u_i e^{j\omega_i t}$. Thus

$$y(t) = \sum_{i=1}^{N} H(j\omega_i) u_i e^{j\omega_i t}.$$

Following the similar method taken in a), we have

$$\begin{aligned} y'(t)y(t) &= \sum_{i=1}^{N} u_i^T H'(j\omega_i) e^{-j\omega_i t} \sum_{k=1}^{N} H(j\omega_k) u_k e^{j\omega_k t} \\ &= \sum_{i=1}^{N}\sum_{k=1}^{N} e^{j(\omega_k - \omega_i)t} u_i^T H'(j\omega_i) H(j\omega_k) u_k. \end{aligned}$$

Thus, P_y can be computed as follows:

$$\begin{aligned}
P_y &= \lim_{L\to\infty} \frac{1}{2L} \sum_i \sum_k u_i^T H'(j\omega_i) H(j\omega_k) u_k \int_{-L}^{L} e^{j(\omega_k-\omega_i)t} dt \\
&= \lim_{L\to\infty} \frac{1}{2L} \sum_i \|H(j\omega_i)u_i\|^2 (2L). \\
\therefore P_y &= \sum_{i=1}^{N} \|H(j\omega_i)u_i\|^2.
\end{aligned}$$

c)Using the fact shown in b),

$$\begin{aligned}
P_y &= \sum_{i=1}^{N} \|H(j\omega_i)u_i\|^2 \\
&\le \sum_{i=1}^{N} \sigma_{max}^2(H(j\omega_i))\|u_i\|^2 \\
&\le \max_i \sigma_{max}^2(H(j\omega_i)) \sum_{i=1}^{N} \|u_i\|^2 \\
&= \max_i \sigma_{max}^2(H(j\omega_i)) P_u \\
\to P_y &\le \max_i \sigma_{max}^2(H(j\omega_i)) P_u \\
P_y &\le \sup_\omega \sigma_{max}^2(H(j\omega)) P_u. \\
\therefore \sup_{P_u=1} P_y &= \|H\|_\infty^2.
\end{aligned}$$

d)Now we have to find an input $u \in \mathbf{X}$ such that $P_y = \|H\|_\infty^2 P_u$. Consider a SVD of $H(j\omega_0)$:

$$H(j\omega_0) = U\Sigma V' = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \begin{pmatrix} - & v_1' & - \\ & \vdots & \\ - & v_n' & - \end{pmatrix}.$$

Let $u = v_1 e^{j\omega_0}$ where ω_0 is such that $\|H\|_\infty = \sigma_{max}(H(j\omega_0))$, then

$$\begin{aligned}
P_y &= \|H(j\omega_0)v_1 e^{j\omega_0}\|_2^2 = \|H(j\omega_0)v_1\|_2^2 = \sigma_1^2\|u_1\|_2^2. \\
\therefore P_y &= \sigma_{max}^2(H(j\omega_0)).
\end{aligned}$$

Indeed, the equality can be achieved by the choice of $u = v_1 e^{j\omega_0}$.

Exercise 16.3 We can restrict our attention to the SISO system since one can prove the MIMO case with similar arguments and use of the SVD.

i.) Input l_∞ Output l_∞
this case was treated in chapter 16.2 of the notes
ii.) Input l_2 Output l_2
this case was treated in chapter 16.3 of the notes
iii.) Input Power Output Power
this case was treated in the Exercise 16.1. Please note that $P_y = \|H\|_\infty^2 P_u$, the given entry in the table corresponds to the rms values.

iv.) Input Power Output l_2
a finite power input normally produces a finite power output (unless the gain of the system is zero at all frequencies) and in that case the 2-norm of the output is infinite.
v.) Input l_2 Output Power
This is now the reveresed situation, but with the same reasoning. A finite energy input produces finite energy output, which has zero power.

vi.) Input Power Output l_∞
Here the idea is that a finite power input can produce a finite power output whose ∞-norm is unbounded. Thinking along the lines of example 15.2 consider the signal $u = \sum_{m=1}^{\infty} v_m(t)$ where $v_m(t) = m$ if $m < t < m + m^{-3}$ and otherwise 0. This signal has finite power and becomes unbounded over time. Take that signal as the input to an LTI system that is just a constant gain.

vii.) Input l_2 Output l_∞

$$\begin{aligned}|y(t)| &= \left|\int_{-\infty}^{\infty} h(t-s)u(s)\ ds\right| \\ |y(t)| &= |< h(t-s), u(s) >| \leq \|h\|_2 \|u\|_2\end{aligned}$$

The last step comes from the application of the Cauchy Schwartz inequality. Taking the sup on the left hand side gives $\|y\|_\infty \leq \|h\|_2\|u\|_2$; now to achieve the bound apply the input $u(t) = h(-t)/\|h\|_2$.

viii.) Input l_∞ Output l_2
Apply a sinusoidal input of unit amplitude such that $j\omega$ is not a zero of the frequency response of the transfer function $H(j\omega)$.

ix.) Input l_∞ Output Power
note that $\{u : \|u\|_\infty \leq 1\}$ is a subset of $\{u : P_u \leq 1\}$. Therefore

$$\sup\{P_y : \|u\|_\infty \leq 1\} \leq \sup\{P_y : P_u \leq 1\};$$

we use the lower bound from case iii.). Note that the entry in the table corresponds to rms and not to power.