MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Spring 2011

Homework 9 Solutions

Exercise 21.1 We can use additive perturbation model with matrices W and Δ given in Figure 21.1.

$$W = \begin{bmatrix} W_{21} & 0 \\ 0 & W_{12} \end{bmatrix},\ \Delta = \begin{bmatrix} 0 & \Delta_1 \\ \Delta_2 & 0 \end{bmatrix},\ P_0 = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix},\ K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

Calculating transfer function from the output of Δ block w to its input z we get

$$M = \begin{bmatrix} 0 & \frac{W_{21}K_1\Delta_1}{1+K_{11}P_{11}} \\ \frac{W_{12}K_2\Delta_2}{1+K_{22}P_{22}} & 0 \end{bmatrix}$$

By assumption in the problem statement the decoupled system is stable, therefore the perturbed system will be stable if $I - M\Delta$ does not have zeros in the closed RHP for any Δ such that $\|\Delta_1\|_\infty \leq 1$ and $\|\Delta_2\|_\infty \leq 1$. By continuity argument this will be true if $I - M\Delta$ does not have zeros on $j\omega$ axis, or equivalently $|det\,(I - M(j\omega)\Delta(j\omega))| > 0$. Let us calculate the determinant in question:

$$\det\,(I - M\Delta) = \det \begin{bmatrix} 1 & \frac{W_{21}K_1\Delta_1}{1+K_{11}P_{11}} \\ \frac{W_{12}K_2\Delta_2}{1+K_{22}P_{22}} & 1 \end{bmatrix} = 1 - \frac{W_{12}W_{21}K_1K_2\Delta_1\Delta_2}{(1+K_{11}P_{11})\,(1+K_{22}P_{22})}$$

To have a stable perturbed system for arbitrary $\|\Delta_1\|_\infty \leq 1$ and $\|\Delta_2\|_\infty \leq 1$ it is necessary and sufficient to impose

$$\left| \frac{W_{12}W_{21}K_1K_2}{(1+K_{11}P_{11})\,(1+K_{22}P_{22})} \right| < 1$$

Since uncertainty blocks enter as a product, the answer will not change if only one block is perturbed.

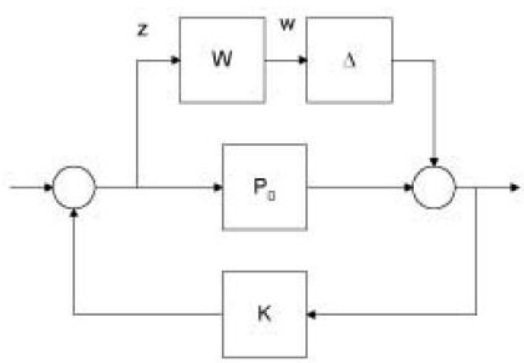

Figure 21.1

Exercise 21.2

According to the lecture notes, we can state the problem equivalently as:

$$\mu(M) = \frac{1}{\inf_{\vec{\delta}\in\mathbb{R}^n}\{\max_i |\delta_i| : \sum_i \delta_i c_i = 1\}}$$

where $c_i = a_i \bar{b}_i$ and $M = \vec{a}\vec{b}'$.

This problem is substantially more difficult because the determination of $\inf_{\vec{\delta}\in\mathbb{R}^n}\{\max_i |\delta_i| : \sum_i \delta_i c_i = 1\}$ is equivalent to:

$$\inf_{\vec{\delta}\in\mathbb{R}^n}\{\max_i |\delta_i| : \Gamma\vec{\delta} = \begin{bmatrix}1\\0\end{bmatrix}\}$$

where:

$$\Gamma = \begin{bmatrix}\mathcal{R}(\vec{c})' \\ \mathcal{I}(\vec{c})'\end{bmatrix}$$

From this point on, we proceed for $n = 3$. The generelization to higher n is immediate.

Notice that the equality $\Gamma\vec{\delta} = \begin{bmatrix}1\\0\end{bmatrix}$ defines a line, say $L_0 \subset \mathbb{R}^3$, in $\mathbb{R}^3$. This makes our problem equivalent to finding the smallest β such that the cube $B_\beta = \{\vec{\delta} \in \mathbb{R}^n : \max_i |\delta_i| \leq \beta\}$ touches the line L_0. That can be done by looking at the following projections:

For every j such that $\mathcal{I}(c_j) \neq 0$, do the following: Look for the smallest β_j such that $\max_i |\delta_i| = \beta_j$ and

$$\sum_i \delta_i(\mathcal{R}(c_i) - \alpha_j \mathcal{I}(c_i)) = 1$$

where $\alpha_j = \frac{\mathcal{R}(c_j)}{\mathcal{I}(c_j)}$. For each such j, the example done in the lecture notes shows that the minimum β_j is $\frac{1}{\sum_i |\mathcal{R}(c_i)-\alpha_j\mathcal{I}(c_i)|}$ and the optimal $\vec{\delta}$ is $\delta_i = \beta_j sgn(\mathcal{R}(c_i) - \alpha_j\mathcal{I}(c_i))$. By doing that, we found the smallest β_j such that the projection of B_{β_j} in $\delta_j = 0$ touches the projection of L_0. Among all of the above candidate solutions, the only admissible is j^* such that $\delta_{j^*} = \frac{-\sum_{i\neq j^*}\mathcal{I}(c_i)\delta_i}{c_{j^*}}$ satisfies $|\delta_{j^*}| \leq \beta_{j^*}$.

The final solution is:

$$\mu(M) = \frac{1}{\beta_{j^*}} = \sum_i |\mathcal{R}(c_i) - \alpha_{j^*}\mathcal{I}(c_i)|$$

Exercise 21.3 The perturbed system can be represented by the diagram in Figure 21.3. The closed loop transfer function from reference input to output is

$$H(s) = \frac{N(s)}{D(s) + K(s)N(s)}$$

The system is stable if the denominator does not have zeros in the closed RHP.

$$D + KN = D_0 + KN_0 + D'_\delta\delta + KN'_\delta\delta = (D_0 + KN_0)\left(1 + \frac{D'_\delta\delta + KN'_\delta\delta}{D_0 + KN_0}\right)$$

From the statement of the problem we know that K is a stabilizing controller, which means that $D_0 + KN_0$ does not have zeros in the closed RHP. Therefore it is sufficient to show that the second parenthesis in the product above does not have zeros in the right half plane. By continuity argument

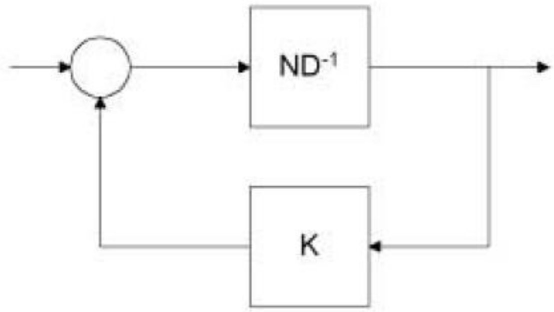

Figure 21.3

we can see that the minimum norm δ at which stability is lost puts at least one root on $j\omega$ axis. Therefore we can rewrite the problem in the following way:

$$\inf_{\omega} \min_{\frac{D'_\delta \delta + K N'_\delta \delta}{D_0 + K N_0}(j\omega) = -1} \|\delta\|_2$$

We can expand the constraint expression, taking into account that the real part has to be equal to -1 and imaginary part ot zero.

$$\inf_{\omega} \min_{A'\delta = b} \|\delta\|_2$$

where

$$A' = \left[\begin{array}{c} Re\left(\frac{D'_\delta + K N'_\delta}{D_0 + K N_0}(j\omega)\right) \\ Im\left(\frac{D'_\delta + K N'_\delta}{D_0 + K N_0}(j\omega)\right) \end{array} \right], \; b = \left[\begin{array}{c} -1 \\ 0 \end{array} \right]$$

The above represents underdetermined least squares problem. For all ω such that rank $A(\omega)$ is 2, the solution is

$$\delta(\omega) = A'(A'A)^{-1}b$$

The 2-norm of this expression can be minimized over ω, and compared to the solutions (if any) with rank$(A(\omega)) < 2$.

Exercise 22.3 a) The modal test is the most convenient in this case. The system is reachable if and only if rank $[\lambda I - A|B] = 5\ \forall\lambda$ (need to check for λ equal to the eigenvalues of A). Observe that when $\lambda = 2$, $[\lambda I - A|B]$ is

$$\begin{bmatrix} 0 & -1 & & & & b_1 \\ & 0 & & & & b_2 \\ & & 0 & & & b_3 \\ & & & -1 & -1 & b_4 \\ & & & & -1 & b_5 \end{bmatrix},$$

which has rank 5 if and only if b_2 and b_3 are linearly independent. Similarly, $\lambda = 3$, $[\lambda I - A|B]$ is

$$\begin{bmatrix} 1 & 1 & & & & b_1 \\ & 1 & & & & b_2 \\ & & 1 & & & b_3 \\ & & & 0 & 1 & b_4 \\ & & & & 0 & b_5 \end{bmatrix},$$

which has rank 5 if and only if $b_5 \neq 0$.

(b) Suppose that $A \in \mathbb{R}^{n\times n}$ has k Jordan blocks of dimensions (number or rows) r_1, r_2, $\ldots$, r_k. Then we must have that $b_{r_1}, b_{r_1+r_2}, \ldots b_{r_1+r_2+r_3+\cdots+r_k} \neq 0$. Furthermore, if blocks r_i and r_j have the same eigenvalue, $b_{r_1+r_2+r_3+\cdots+r_i}$ and $b_{r_1+r_2+r_3+\cdots+r_j}$ must be linearly independent. These conditions imply that the input can excite the beginning of each Jordan chain, and hence has an impact on each of the states.

(c) If the $b_{i's}$ are scalars, then they are linearly dependent (multiples of each other), so if two of the Jordan blocks have the same eigenvalues the rank of $[\lambda I - A|B]$ is less than n.

Alternatively,

a) The system is reachable if none of the left eigenvectors of matrix A are orthogonal to B. Notice that to control the states corresponding to a Jordan block, it is sufficient to excite only the state corresponding the beginning of the Jordan chain, or the last element in the Jordan block (convince yourself of this considering a DT system for example). Thus it is not necessary that generalized eigenvectors are not orthogonal to the B matrix! Besides, notice that if two or more Jordan blocks have the same eigenvalue than any linear combination of eigenvectors corresponding to those Jordan blocks is a left eigenvector again. In case (a) we can identify left eigenvectors of matrix A:

$$\begin{aligned} w_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}' \\ w_3 &= \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \end{bmatrix}' \\ w_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}' \end{aligned}$$

Any linear combination of w_2 and w_3 is also a left eigenvector. We can see that $w_k' B = b_k'$ - k^{th} row of matrix B. Therefore for reachability of matrix A we need to have at least one non-zero element in 5^{th} row and linear independence of 2^{rd} and 3^{rd} rows of matrix B.

b) Generalizing to an arbitrary matrix in Jordan form we can see that all rows of matrix B corresponding to a Jordan block with unique eigenvalue should have at least one non-zero element, and rows corresponding to Jordan blocks with repeated eigenvalues should be linearly independent.

c) If there are two or more Jordan blocks then we can find a linear combination of the eigenvectors which is orthogonal to the vector b, since two real numbers are obviously linearly dependent.

Exercise 22.4 The open loop system is reachable and has a closed-loop expression as follows:

$$x_{k+1} = Ax_k + B(w_k + f(x_k)),$$

where $f(\cdot)$ is an arbitrary but *known* function. Since the open loop system is reachable, there exists the control input u^* such that

$$u^* = \begin{pmatrix} u^*(0) & \cdots & u^*(n-1) \end{pmatrix}^T.$$

that can drive the system to a target state in $\mathbb{R}^n$, $x_f = x^*(n)$. Thus let's define a trajectory $x^*(k)$ such that it starts from the origin and gets to $x^*(n)$ by the control input u^*. Then, since $u(k) = w(k) + f(x(k))$, let $w(k) = u^*(k) - f(x^*(K))$. Then this $w(k)$ can always take the system state from the origin to any specified target state in no more than n steps.