Massachusetts Institute of Technology

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Lecture 5: Lyapunov Functions and Storage Functions
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This lecture gives an introduction into system analysis using Lyapunov functions and their generalizations.

5.1 Recognizing Lyapunov functions

There exists a number of slightly different ways of defining what constitutes a Lyapunov function for a given system. Depending on the strength of the assumptions, a variety of conclusions about a system's behavior can be drawn.

5.1.1 Abstract Lyapunov and storage functions

In general, *Lyapunov functions* are real-valued functions of system's state which are monotonically non-increasing on every signal from the system's behavior set. More generally, *stotage functions* are real-valued functions of system's state for which explicit upper bounds of increments are available.

Let $\mathcal{B} = \{z\}$ be a behavior set of a system (i.e. elements of $\mathcal{B}$ are are vector signals, which represent all possible outputs for autonomous systems, and all possible input/output pairs for systems with an input). Remember that by a *state* of a system we mean a function $x: \mathcal{B} \times [0, \infty) \mapsto X$ such that two signals $z_1, z_2 \in \mathcal{B}$ define same state of $\mathcal{B}$ at time t whenever $x(z_1(\cdot), t) = x(z_2(\cdot), t)$ (see Lecture 1 notes for details and examples). Here X is a set which can be called the *state space* of $\mathcal{B}$. Note that, given the behavior set $\mathcal{B}$, state space X is not uniquelly defined.

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Definition A real-valued function $V: X \mapsto \mathbf{R}$ defined on state space X of a system with behavior set $\mathcal{B}$ and state $x: \mathcal{B} \times [0,\infty) \mapsto X$ is called a *Lyapunov function* if $t \mapsto V(t) = V(x(t)) = V(x(z(\cdot),t))$ is a non-increasing function of time for every $z \in \mathcal{B}$.

According to this definition, Lyapunov functions provide limited but very explicit information about system behavior. For example, if $X = \mathbf{R}^n$ and $V(x(t)) = |x(t)|^2$ is a Lyapunov function then we now that system state $x(t)$ remains bounded for all times, though we may have no idea of what the exact value of $x(t)$ is.

For conservative systems in physics, the total energy is always a Lyapunov function. Even for non-conservative systems, it is frequently important to look for energy-like expressions as Lyapunov function candidates.

One can say that Lyapunov functions have an *explicit* upper bound (zero) imposed on their increments along system trajectories:

$$V(x(z(\cdot),t_1)) - V(x(z(\cdot),t_0)) \leq 0 \;\; \forall\; t_1 \geq t_0 \geq 0, \;\; z \in \mathcal{B}.$$

A useful generalization of this is given by *storage functions*.

Definition Let $\mathcal{B}$ be a set of n-dimensional vector signals $z: [0,\infty) \mapsto \mathbf{R}^n$. Let $\sigma: \mathbf{R}^n \mapsto \mathbf{R}$ be a given function such that $\sigma(z(t))$ is locally integrable for all $z(\cdot) \in \mathcal{B}$. A real-valued function $V: X \mapsto \mathbf{R}$ defined on state space X of a system with behavior set $\mathcal{B}$ and state $x: \mathcal{B} \times [0,\infty) \mapsto X$ is called a *storage function with supply rate* σ if

$$V(x(z(\cdot),t_1)) - V(x(z(\cdot),t_0)) \leq \int_{t_0}^{t_1} \sigma(z(t))dt \;\; \forall\; t_1 \geq t_0 \geq 0, \;\; z \in \mathcal{B}. \tag{5.1}$$

In many applications σ is a function comparing the instantaneous values of input and output. For example, if $\mathcal{B} = \{z(t) = [v(t); w(t)]\}$ is the set of all possible input/output pairs of a given system, existence of a non-negative storage function with supply rate $\sigma(z(t)) = |v(t)|^2 - |w(t)|^2$ proves that *power* of the output, defined as

$$\|w(\cdot)\|_p^2 = \lim_{T\to\infty} \sup_{t \leq T} \frac{1}{t} \int_0^t |w(\tau)|^2 d\tau,$$

never exceed power of the input.

Example 5.1 Let behavior set $\mathcal{B} = \{(i(t), v(t))\}$ descrive the (dynamcal) voltage-current relation of a *passive* single port electronic circuit. Then the total energy $E = E(t)$ accumulated in the circuit can serve as a storage function with supply rate

$$\sigma(i(t), v(t)) = i(t)v(t).$$

5.1.2 Lyapunov functions for ODE models

It is important to have tools for verifying that a given function of a system's state is monotonically non-increasing along system trajectories, without explicitly calculating solutions of system equations. For systems defined by ODE models, this can usually be done.

Consider an autonomous system defined by ODE model

$$\dot{x}(t) = a(x(t)), \tag{5.2}$$

where $a: X \mapsto \mathbf{R}^n$ is a function defined on a subset of $\mathbf{R}^n$. A functional $V: X \mapsto \mathbf{R}$ is a Lyapunov function for system (5.2) if $t \mapsto V(x(t))$ is monotonically non-increasing for every solution of (5.2). Remember that $x: [t_0, t_1] \to X$ is called a solution of (5.2) if the composition $a \circ x$ is absolutely integrable on $[t_0, t_1]$ and equality

$$x(t) = x(t_0) + \int_{t_0}^{t} a(x(\tau))d\tau$$

holds for all $t \in [t_0, t_1]$.

To check that a given function V is a Lyapunov function for system (5.2), one usually attempts to differentiate $V(x(t))$ with respect to t. If X is an open set, and both V and x are differentiable (note that the differentiability of x is assured by the *continuity* of a), the composition $t \mapsto V(x(t))$ is also differentiable, and the monotonicity condition is given by

$$\nabla V(\bar{x})a(\bar{x}) \leq 0 \quad \forall\, \bar{x} \in X, \tag{5.3}$$

where $\nabla V(x)$ denotes the gradient of V at x.

In some applications one may be forced to work with systems that have non-differentiable solutions (for example, because of a jump in an external input signal). The convenient Lyapunov function candidates V may also be non-differentiable at some points. In such situations, it is tempting to consider, for every $\bar{x} \in X$, the subgradient of V at $\bar{x} \in X$ in the direction $a(\bar{x})$. One may expect that non-positivity of such subgradients, which can be expressed as

$$\lim_{\epsilon \to 0, \epsilon > 0} \sup_{0 < t < \epsilon} \frac{V(\bar{x} + ta(\bar{x})) - V(\bar{x})}{t} \leq 0 \quad \forall\, \bar{x} \in X, \tag{5.4}$$

implies that V is a valid Lyapunov function. However, this is not always true.

Example 5.2 Using the famous example of a Kantor function, one can construct a bounded integrable function $a: \mathbf{R} \mapsto \mathbf{R}$ and a continuous function $V: \mathbf{R} \mapsto \mathbf{R}$ such that $t \mapsto V(\bar{x} + ta(\bar{x}))$ is *constant* in a neigborhood of $t = 0$ for every $\bar{x} \in \mathbf{R}$, but the ODE (5.2) has a solution for which $V(x(t))$ is strictly monotonically increasing!

Here by a Kantor function we mean a *continuous* strictly monotonic function $k: [0, 1] \mapsto \mathbf{R}$ such that $k(0) = 0$ and $k(1) = 1$ despite the fact that $k(t)$ is constant on a family $\mathcal{T} = \{T\}$ of open disjoint intervals $T \subset [0,1]$ of total length 1. Indeed, for a fixed Kantor function k define

$$V(\bar{x}) = \text{floor}(\bar{x}) + k(1 - \text{floor}(\bar{x})),$$

where $\text{floor}(\bar{x})$ denotes the largest integer not larger than $\bar{x}$. Let $a(\bar{x})$ be zero on every interval $(m + t_1, m + t_2)$, where m is an integer and $(t_1, t_2) \in \mathcal{T}$, and $a(\bar{x}) = 0$ otherwise. Then $x(t) \equiv t$ is a solution of ODE (5.2), but $V(x(t))$ is strictly monotonically increasing, despite the fact that $t \mapsto V(\bar{x} + ta(\bar{x}))$ is *constant* in a neigborhood of $t = 0$ for every $\bar{x} \in \mathbf{R}$.

However, if V and all solutions of (5.2) are "smooth enough", condition (5.4) is sufficient for V to be a Lyapunov function.

Theorem 5.1 *If X is an open set in $\mathbf{R}^n$, $V: X \mapsto \mathbf{R}$ is locally Lipschitz, $a: X \mapsto \mathbf{R}^n$ is continuous, and condition (5.4) is satisfied then $V(x(t))$ is monotonically non-increasing for all solutions $x: [t_0, t_1] \mapsto X$ of (5.2).*

Proof We will use the following statement: *if $h: [t_0, t_1] \mapsto \mathbf{R}$ is continuous and satisfies*

$$\lim_{d \to 0, d>0} \sup_{\delta \in (0,d)} \frac{h(t+\delta) - h(t)}{\delta} \le 0 \quad \forall\, t \in [t_0, t_1), \tag{5.5}$$

then h is monotonically non-increasing. Indeed, for every $r > 0$ let $h_r(t) = h(t) - rt$. If h_r is monotonically non-increasing for all $r > 0$ then so is h. Otherwise, assume that $h_r(t_3) > h_r(t_2)$ for some $t_0 \le t_2 < t_3 \le t_1$ and $r > 0$. Let t_4 be the maximal solution of equation $h_r(t) = h_r(t_2)$ with $t \in [t_2, t_3]$. Then $h_r(t) > h_r(t_4)$ for all $t \in (t_4, t_3]$, and hence (5.5) is violated at $t = t_4$.

Now let M be the Lipschitz constant for V in a neigborhood of the trajectory of x. Since a is continuous,

$$\lim_{\delta \to 0, \delta > 0} \left| \frac{x(t+\delta) - x(t)}{\delta} - a(x(t)) \right| = 0 \quad \forall\, t.$$

Hence the maximum (over $t \in [t_0, t_1 - \delta]$) of

$$\frac{V(x(t+\delta)) - V(x(t))}{\delta} = \frac{V(x(t) + \delta a(x(t))) - V(x(t))}{\delta} + \frac{V(x(t+\delta)) - V(x(t) + \delta a(x(t)))}{\delta}$$

$$\le \frac{V(x(t) + \delta a(x(t))) - V(x(t))}{\delta} + M \left| \frac{x(t+\delta) - x(t) - \delta a(x(t))}{\delta} \right|$$

converges to a non-positive limit as $\delta \to 0$. ■

A time-varying ODE model

$$\dot{x}_1(t) = a_1(x_1(t), t) \tag{5.6}$$

can be converted to (5.2) by introducing

$$x(t) = [x_1(t); t], \quad a([barx; \tau]) = [a_1(\bar{x}, \tau); 1],$$

in which case the Lyapunov function $V = V(x(t)) = V(x_1(t), t)$ can naturally depend on time.

5.1.3 Storage functions for ODE models

Consider the ODE model

$$\dot{x}(t) = f(x(t), u(t)) \tag{5.7}$$

with state vector $x(t) \in X \subset \mathbf{R}^n$, input $u(t) \in U \subset \mathbf{R}^m$, where $f: X \times U \mapsto \mathbf{R}^n$ is a given function. Let $\sigma: X \times U \mapsto \mathbf{R}$ be a given functional. A function $V: X \mapsto \mathbf{R}$ is called a *storage function with supply rate σ for system* (5.7)

$$V(x(t_1)) - V(x(t_0)) \le \int_{t_0}^{t_1} \sigma(x(t), u(t)) dt$$

for every pair of integrable functions $x: [t_0, t_1] \mapsto X$, $u: [t_0, t_1] \mapsto U$ such that the composition $t \mapsto f(x(t), u(t))$ satisfies the identity

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(t), u(t)) dt$$

for all $t \in [t_0, t_1]$.

When X is an open set, f and σ are continuous, and V is continuously differentiable, verifying that a given f is a valid storage function with supply rate σ is straightforward: it is sufficient to check that

$$\nabla V \cdot f(\bar{x}, \bar{u}) \le \sigma(\bar{x}, \bar{u}) \ \ \forall \ \bar{x} \in X, \bar{u} \in U.$$

When V is locally Lipschitz, the following generalization of Theorem 5.1 is available.

Theorem 5.2 *If X is an open set in $\mathbf{R}^n$, $V: X \mapsto \mathbf{R}$ is locally Lipschitz, $f, \sigma: X \times U \mapsto \mathbf{R}^n$ are continuous, and condition*

$$\lim_{\epsilon \to 0, \epsilon > 0} \sup_{0 < t < \epsilon} \frac{V(\bar{x} + t f(\bar{x}, \bar{u})) - V(\bar{x})}{t} \le \sigma(\bar{x}, \bar{u}) \ \ \forall \ \bar{x} \in X, \bar{u} \in U \tag{5.8}$$

is satisfied then $V(x(t))$ is a storage function with supply rate σ for system (5.7).

The proof of the theorem follows the lines of Theorem 5.1. Further generalizations to discontinuous functions f, etc., are possible.