Lecture 7: Finding Lyapunov Functions[1]

This lecture gives an introduction into basic methods for finding Lyapunov functions and storage functions for given dynamical systems.

7.1 Convex search for storage functions

The set of all real-valued functions of system state which do not increase along system trajectories is *convex*, i.e. closed under the operations of addition and multiplication by a positive constant. This serves as a basis for a general procedure of searching for Lyapunov functions or storage functions.

7.1.1 Linearly parameterized storage function candidates

Consider a system model given by discrete time state space equations

$$x(t+1) = f(x(t), w(t)), \quad y(t) = g(x(t), w(t)), \tag{7.1}$$

where $x(t) \in X \subset \mathbf{R}^n$ is the system state, $w(t) \in W \subset \mathbf{R}^m$ is system input, $y(t) \in Y \subset \mathbf{R}^k$ is system output, and $f : X \times W \mapsto X$, $g : X \times W \mapsto Y$ are given functions. A functional $V : X \mapsto \mathbf{R}$ is a *storage function* for system (7.1) with *supply rate* $\sigma : Y \times W \mapsto \mathbf{R}$ if

$$V(x(t+1)) - V(x(t)) \le \sigma(y(t)) \tag{7.2}$$

for every solution of (7.1), i.e. if

$$V(f(\bar{x}, \bar{w})) - V(\bar{x}) \le \sigma(g(\bar{x}, \bar{w}), \bar{w}) \quad \forall \, \bar{x} \in X, \; \bar{w} \in W. \tag{7.3}$$

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In particular, when $\sigma \equiv 0$, this yields the definition of a Lyapunov function.

Finding, for a given supply rate, a valid storage function (or at least proving that one exists) is a major challenge in constructive analysis of nonlinear systems. The most common approach is based on considering a *linearly parameterized subset of storage function candidates* $\mathcal{V}$ defined by

$$\mathcal{V} = \{V(\bar{x}) = \sum_{q=1}^{N} \tau_q V_q(\bar{x}), \tag{7.4}$$

where $\{V_q\}$ is a fixed set of basis functions, and τ_k are parameters to be determined. Here every element of $\mathcal{V}$ is considered as a storage function *candidate*, and one wants to set up an efficient search for the values of τ_k which yield a function V satisfying (7.3).

Example 7.1 Consider the finite state automata defined by equations (7.1) with value sets

$$X = \{1, 2, 3\}, \quad W = \{0, 1\}, \quad Y = \{0, 1\},$$

and with dynamics defined by

$$f(1, 1) = 2, \ f(2, 1) = 3, \ f(3, 1) = 1, \ f(1, 0) = 1, \ f(2, 0) = 2, \ f(3, 0) = 2,$$

$$g(1, 1) = 1, \ g(\bar{x}, \bar{w}) = 0 \ \forall \ (\bar{x}, \bar{w}) \neq (1, 1).$$

In order to show that the amount of 1's in the output is never much larger than one third of the amount of 1's in the input, one can try to find a storage function V with supply rate

$$\sigma(\bar{y}, \bar{w}) = \bar{w} - 3\bar{y}.$$

Taking three basis functions V_1, V_2, V_3 defined by

$$V_k(\bar{x}) = \begin{cases} 1, & \bar{x} = k, \\ 0, & \bar{x} \neq k, \end{cases}$$

the conditions imposed on τ_1, τ_2, τ_3 can be written as the set of six affine inequalities (7.3), two of which (with $(\bar{x}, \bar{w}) = (1, 0)$ and $(\bar{x}, \bar{w}) = (2, 0)$) will be satisfied automatically, while the other four are

$$\tau_2 - \tau_3 \leq 1 \quad \text{at } (\bar{x}, \bar{w}) = (3, 0),$$
$$\tau_2 - \tau_1 \leq -2 \quad \text{at } (\bar{x}, \bar{w}) = (1, 1),$$
$$\tau_3 - \tau_2 \leq 1 \quad \text{at } (\bar{x}, \bar{w}) = (2, 1),$$
$$\tau_1 - \tau_3 \leq 1 \quad \text{at } (\bar{x}, \bar{w}) = (3, 1).$$

Solutions of this linear program are given by

$$\tau_1 = c, \quad \tau_2 = c - 2, \quad \tau_3 = c - 1,$$

where $c \in \mathbf{R}$ is arbitrary. It is customary to normalize storage and Lyapunov functions so that their minimum equals zero, which yields $c = 2$ and

$$\tau_1 = 2, \ \tau_2 = 0, \ \tau_3 = 1.$$

Now, summing the inequalities (7.2) from $t = 0$ to $t = T$ yields

$$3 \sum_{t=0}^{T-1} y(t) \leq V(x(0)) - V(x(T)) + \sum_{t=0}^{T-1} w(t),$$

which is implies the desired relation between the numbers of 1's in the input and in the output, since $V(x(0)) - V(x(T))$ cannot be larger than 2.

7.1.2   Storage functions via cutting plane algorithms

The possibility to reduce the search for a valid storage function to convex optimization, as demonstrated by the example above, is a general trend. One general situation in which an efficient search for a storage function can be performed is when a cheap procedure of checking condition (7.3) (an *oracle*) is available.

Assume that for every given element $V \in \mathcal{V}$ it is possible to find out whether condition (7.3) is satisfied, and, in the case when the answer is negative, to produce a pair of vectors $\bar{x} \in X$, $\bar{w} \in W$ for which the inequality in (7.3) does not hold. Select a sufficiently large set T_0 (a polytope or an ellipsoid) in the space of parameter vector $\tau = (\tau_q)_{q=1}^N$ (this set will limit the search for a valid storage function). Let τ^* be the "center" of T_0. Define V by the τ^*, and apply the verification "oracle" to it. If V is a valid storage function, the search for storage function ends successfully. Otherwise, the "invalidity certificate" $(\bar{x}, \bar{w})$ produced by the oracle yields a hyperplane separating τ^* and the (unknown) set of τ defining valid storage functions, thus cutting a substantial portion from the search set T_0, reducing it to a smaller set T_1. Now re-define τ^* as the center of T_1 and repeat the process by constructing a sequence of monotonically decreasing search sets T_k, until either a valid storage function is found, or T_k shrinks to nothing.

With an appropriate selection of a class of search sets T_k (ellipsoids or polytopes are most frequently used) and with an adequate definition of a "center" (the so-called "analytical center" is used for polytopes), the volume of T_k can be made exponentially decreasing, which constitutes fast convergence of the search algorithm.

7.1.3   Completion of squares

The success of the search procedure described in the previous section depends heavily on the choice of the basis functions V_k. A major difficulty to overcome is verification of (7.3) for a given V. It turns out that the only known large linear space of functionals

$F : \mathbf{R}^n \mapsto \mathbf{R}$ which admits efficient check of non-negativity of its elements is the set of *quadratic forms*

$$F(\bar{x}) = \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix}' Q \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix}, \quad (Q = Q')$$

for which nonnegativity is equivalent to positive semidefiniteness of the coefficient matrix Q.

This observation is exploited in the linear-quadratic case, when f, g are linear functions

$$f(\bar{x}, \bar{w}) = A\bar{x} + B\bar{w}, \quad g(\bar{x}, \bar{w}) = C\bar{x} + D\bar{w},$$

and σ is a quadratic form

$$\sigma(\bar{x}, \bar{w}) = \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}' \Sigma \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}.$$

Then it is natural to consider *quadratic* storage function candidates

$$V(\bar{x}) = \bar{x}' P \bar{x}$$

only, and (7.3) transforms into the (symmetric) *matrix inequality*

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} \le \Sigma. \tag{7.5}$$

Since this inequality is *linear* with respect to its parameters P and Σ, it can be solved relatively efficiently even when additional linear constraints are imposed on P and Σ.

Note that a quadratic functional is non-negative if and only if it can be represented as a sum of squares of linear functionals. The idea of checking non-negativity of a functional by trying to represent it as a sum of squares of functions from a given linear set can be used in searching for storage functions of general nonlinear systems as well. Indeed, let $\hat{H} : \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^M$ and $\hat{V} : \mathbf{R}^n \mapsto \mathbf{R}^N$ be arbitrary vector-valued functions. For every $\tau \in \mathbf{R}^N$, condition (7.3) with

$$V(\bar{x}) = \tau' \hat{V}(\bar{x})$$

is implied by the identity

$$\tau' \hat{V}(f(\bar{x}, \bar{w})) - \tau' \hat{V}(\bar{x}) + \hat{H}(\bar{x}, \bar{w})' S \hat{H}(\bar{x}, \bar{w}) = \sigma(\bar{x}, \bar{w}) \quad \forall \ \bar{x} \in X, \ \bar{w} \in W, \tag{7.6}$$

as long as $S = S' \ge 0$ is a positive semidefinite symmetric matrix. Note that both the storage function candidate parameter τ and the "sum of squares" parameter $S = S' \ge 0$ enter constraint (7.6) linearly. This, the search for a valid storage function is reduced to *semidefinite program.*

In practice, the scalar components of vector $\hat{H}$ should include enough elements so that identity (7.6) can be achieved for every $\tau \in \mathbf{R}^N$ by choosing an appropriate $S = S'$ (not necessarily positivie semidefinite). For example, if f, g, σ are polynomials, it may be a good idea to use a polynomial $\hat{V}$ and to define $\hat{H}$ as the vector of monomials up to a given degree.

7.2 Storage functions with quadratic supply rates

As described in the previous section, one can search for storage functions by considering linearly parameterized sets of storage function candidates. It turns out that storage functions derived for subsystems of a given system can serve as convenient building blocks (i.e. the components V_q of $\hat{V}$). Indeed, assume that $V_q = V_q(x(t))$ are storage functions with supply rates $\sigma_q = \sigma_q(z(t))$. Typically, $z(t)$ includes $x(t)$ as its component, and has some additional elements, such as inputs, outputs, and othe nonlinear combinations of system states and inputs. If the objective is to find a storage function V_* with a given supply rate σ_*, one can search for V_* in the form

$$V(x(t)) = \sum_{q=1}^{N} V_q(x(t)), \quad \tau_q \geq 0, \tag{7.7}$$

where τ_q are the search parameters. Note that in this case it is known a-priori that every V_* in (7.7) is a storage function with supply rate

$$\sigma(z(t)) = \sum_{q=1}^{N} \tau_q \sigma_q(z(t)). \tag{7.8}$$

Therefore, in order to find a storage function with supply rate $\sigma_* = \sigma_*(z(t))$, it is sufficient to find $\tau_q \geq 0$ such that

$$\sum_{q=1}^{N} \tau_1 \sigma_q(\bar{z}) \leq \sigma_*(\bar{z}) \quad \forall \ \bar{z}. \tag{7.9}$$

When σ_*, σ_q are generic functions, even this simplified task can be difficult. However, in the important special case when σ_* and σ_q are *quadratic functionals*, the search for τ_q in (7.9) becomes a semidefinite program.

In this section, the use of storage functions with quadratic supply rates is discussed.

7.2.1 Storage functions for LTI systems

A quadratic form $V(\bar{x}) = \bar{x}'P\bar{x}$ is a storage function for LTI system

$$\dot{x} = Ax + Bw \tag{7.10}$$

with quadratic supply rate

$$\sigma(\bar{x}, \bar{w}) = \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}' \Sigma \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}$$

if and only if matrix inequality (7.5) is satisfied.

The well-known *Kalman-Popov-Yakubovich Lemma*, or *positive real lemma* gives useful frequency domain condition for *existence* of such $P = P'$ for given A, B, Σ.

Theorem 7.1 *Assume that the pair (A, B) is controllable. A symmetric matrix $P = P'$ satisfying (7.5) exists if and only if*

$$\begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}' \Sigma \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix} \geq 0 \quad \text{whenever} \quad j\omega\bar{x} = A\bar{x} + B\bar{w} \quad \text{for some } \omega \in \mathbf{R}. \tag{7.11}$$

Moreover, if there exists a matrix K such that $A + BK$ is a Hurwitz matrix, and

$$\begin{bmatrix} I \\ K \end{bmatrix}' \Sigma \begin{bmatrix} I \\ K \end{bmatrix} \leq 0,$$

then all such matrices $P = P'$ are positive semidefinite.

Example 7.2 Let $G(s) = C(sI - A)^{-1}B + D$ be a stable transfer function (i.e. matrix A is a Huewitz matrix) with a controllable pair (A, B). Then $|G(j\omega)| \leq 1$ for all $\omega \in \mathbf{R}$ if and only if there exists $P = P' \geq 0$ such that

$$2\bar{x}'P(A\bar{x} + B\bar{w}) \leq |\bar{w}|^2 - |C\bar{x} + D\bar{w}|^2 \quad \forall \, \bar{x} \in \mathbf{R}^n, \; \bar{w} \in \mathbf{R}^m.$$

This can be proven by applying Theorem 7.1 with

$$\sigma(\bar{x}, \bar{w}) = |\bar{w}|^2 - |C\bar{x} + D\bar{w}|^2$$

and $K = 0$.

7.2.2 Storage functions for sector nonlinearities

Whenever two components $v = v(t)$ and $w = w(t)$ of the system trajectory $z = z(t)$ are related in such a way that the pair $(v(t), w(t))$ lies in the cone between the two lines $w = k_1 v$ and $v = k_2 v$, $V \equiv 0$ is a storage function for

$$\sigma(z(t)) = (w(t) - k_1 v(t))(k_2 v(t) - w(t)).$$

For example, if $w(t) = v(t)^3$ then $\sigma(z(t)) = v(t)w(t)$. If $w(t) = \sin(t)\sin(v(t))$ then $\sigma(z(t)) = |v(t)|^2 - |w(t)|^2$.

7.2.3 Storage for scalar memoryless nonlinearity

Whenever two components $v = v(t)$ and $w = w(t)$ of the system trajectory $z = z(t)$ are related by $w(t) = \phi(v(t))$, where $\phi : \mathbf{R} \mapsto \mathbf{R}$ is an integrable function, and $v(t)$ is a component of system state, $V(x(t)) = \psi(v(t))$ is a storage function with supply rate

$$\sigma(z(t)) = \dot{v}(t)w(t),$$

where

$$\psi(y) = \int_0^y \phi(\tau)d\tau.$$

7.3 Implicit storage functions

A number of important results in nonlinear system analysis rely on storage functions for which no explicit formula is known. It is frequently sufficient to provide a lower bound for the storage function (for example, to know that it takes only non-negative values), and to have an analytical expression for the supply rate function σ.

In order to work with such "implicit" storage functions, it is helpful to have theorems which guarantee existence of non-negative storage functions for a given supply rate. In this regard, Theorem 7.1 can be considered as an example of such result, stating existence of a storage function for a linear and time invariant system as an implication of a frequency-dependent matrix inequality. In this section we present a number of such statements which can be applied to nonlinear systems.

7.3.1 Implicit storage functions for abstract systems

Consider a system defined by behavioral set $\mathcal{B} = \{z\}$ of functions $z : [0, \infty) \mapsto \mathbf{R}^q$. As usually, the system can be autonomous, in which case $z(t)$ is the output at time t, or with an input, in which case $z(t) = [v(t); w(t)]$ combines vector input $v(t)$ and vector output $w(t)$.

Theorem 7.2 *Let $\sigma : \mathbf{R}^q \mapsto \mathbf{R}$ be a function and let $\mathcal{B}$ be a behavioral set, consisting of some functions $z : [0, \infty) \mapsto \mathbf{R}^q$. Assume that the composition $\sigma(z(t))$ is integrable over every bounded interval (t_0, t_1) in $\mathbf{R}_+$ for all $z \in \mathcal{B}$. For $t_0, t \in \mathbf{R}_+$ define*

$$\mathcal{I}(z, t_0, t) = \int_{t_0}^{t} \sigma(z(\tau))d\tau.$$

The following conditions are equivalent:

(a) *for every $z_0 \in \mathcal{B}$ and $t_0 \in \mathbf{R}_+$ the set of values $\mathcal{I}(z, t_0, t)$, taken for all $t \geq t_0$ and for all $z \in \mathcal{B}$ defining same state as z_0 at time t_0, is bounded from below;*

(b) *there exists a non-negative storage function $V : \mathcal{B} \times \mathbf{R}_+ \mapsto \mathbf{R}_+$ (such that $V(z_1, t) = V(z_2, t)$ whenever z_1 and z_2 define same state of $\mathcal{B}$ at time t) with supply rate σ.*

Moreover, when condition (a) is satisfied, a storage function V from (b) can be defined by

$$V(z_0(\cdot), t_0) = -\inf \mathcal{I}(z, t_0, t), \tag{7.12}$$

where the infimum is taken over all $t \geq t_0$ and over all $z \in \mathcal{B}$ defining same state as z_0 at time t_0.

Proof Implication (b)$\Rightarrow$(a) follows directly from the definition of a storage function, which requires

$$V(z_0, t_1) - V(z_0, t_0) \leq \mathcal{I}(z, t_0, t_1) \tag{7.13}$$

for $t_1 \geq t_0$, $z_0 \in \mathcal{B}$. Combining this with $V \geq 0$ yields

$$\mathcal{I}(z, t_0, t_1) \geq -V(z, t_0) = -V(z_0, t_0)$$

for all z, z_0 defining same state of $\mathcal{B}$ at time t_0.

Now let us assume that (a) is valid. Then a finite infimum in (7.12) exists (as an infimum over a non-empty set bounded from below) and is not positive (since $\mathcal{I}(z_0, t_0, t_0) = 0$). Hence V is correctly defined and not negative. To finish the proof, let us show that (7.13) holds. Indeed, if z_1 defines same state as z_0 at time t_1 then

$$z_{01}(t) = \begin{cases} z_0(t), & t \leq t_1, \\ z_1(t), & t > t_1 \end{cases}$$

defines same state as z_0 at time $t_0 < t_1$ (explain why). Hence the infimum of $\mathcal{I}(z, t_0, t)$ in the definition of V is not larger than the infimum of integrals of all such z_{01}, over intervals of length not smaller than $t_1 - t_0$. These integrals can in turn be decomposed into two integrals

$$\mathcal{I}(z_{01}, t_0, t) = \mathcal{I}(z_0, t_0, t_1) + \mathcal{I}(z_1, t_1, t),$$

which yields the desired inequality. ∎

7.3.2 Storage functions for ODE models

As an important special case of Theorem 7.2, consider the ODE model

$$\dot{x}(t) = f(x(t), w(t)), \tag{7.14}$$

defined by a function $f : X \times W \mapsto \mathbf{R}^n$, where X, W are subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. Cxonsider the behavior model $\mathcal{B}$ consisting of all functions $z(t) = [x(t); v(t)]$ where $x : [0, \infty) \mapsto X$ is a solution of (7.14). In this case, two signals $z_1 = [x_1; v_1]$ and $z_2 = [x_2; v_2]$ define same state of $\mathcal{B}$ at time t_0 if and only if $x_1(t_0) = x_2(t_0)$. Therefore, according to Theorem 7.2, for a given function $\sigma : X \times W \mapsto \mathbf{R}$, existence of a function $V : X \times \mathbf{R}_+ \mapsto \mathbf{R}_+$ such that

$$V(x(t_2), t_2) - V(x(t_1), t_1) \leq \int_{t_1}^{t_2} \sigma(x(t), v(t)) dt$$

for all $0 \leq t_1 \leq t_2$, $[x; v] \in \mathcal{B}$ is equivalent to finiteness of the infimum of the integrals

$$\int_{t_0}^{t} \sigma(x(\tau), v(\tau)) d\tau$$

over all solutions of (7.14) with a fixed $x(t_0) = \bar{x}_0$ which can be extended to the time interval $[0, \infty)$.

In the case when $X = \mathbf{R}^n$, and $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is such that existence and uniqueness of solutions $x : [0, \infty) \mapsto \mathbf{R}^n$ is guaranteed for all locally integrable inputs $w : [0, \infty) \mapsto W$ and all initial conditions $x(t_0) = \bar{x}_0 \in \mathbf{R}^n$, the infimum in (7.12) (and hence, the corresponding storage function) do not depend on time. If, in addition, f is continuous and V is continuously differentiable, the well-known dynamic programming condition

$$\lim_{\epsilon \to 0, \epsilon > 0} \inf_{\bar{w} \in W, \bar{x} \in B_\epsilon(\bar{x}_0)} \{\sigma(\bar{x}, \bar{w}) - \nabla V(\bar{x}) f(\bar{x}, \bar{w})\} 0 \leq \inf_{\bar{w} \in W} \{\sigma(\bar{x}_0, \bar{w}) - \nabla V(\bar{x}_0) f(\bar{x}_0, \bar{w})\} \ \forall \bar{x}_0 \in \mathbf{R}^n$$
(7.15)

will be satisfied. However, using (7.15) requires a lot of caution in most cases, since, even for very smooth f, σ, the resulting storage function V does not have to be differentiable.

7.3.3 Zames-Falb quadratic supply rate

A non-trivial and powerful case of an implicitly defined storage function with a quadratic supply rate was introduced in late 60-s by G. Zames and P. Falb.

Theorem 7.3 *Let A, B, C be matrices such that A is a Hurwitz matrix, and*

$$\int_0^\infty |Ce^{At}B| dt < 1.$$

Let $\phi : \mathbf{R} \mapsto \mathbf{R}$ be a monotonic odd function such that

$$0 \leq \bar{w}\phi(\bar{w}) \leq |\bar{w}|^2 \ \forall \bar{w} \in \mathbf{R}.$$

Then for all $\theta < 1$ system
$$\dot{x}(t) = Ax(t) + Bw(t)$$

has a non-negative storage function with supply rate
$$\sigma_+(\bar{x}, \bar{w}) = (\bar{w} - \theta\phi(\bar{w}))(\bar{w} - C\bar{x}),$$

and system
$$\dot{x}(t) = Ax(t) + B(w(t) - \theta\phi(w(t)))$$

has a non-negative storage function with supply rate
$$\sigma_-(\bar{x}, \bar{w}) = (\bar{w} - \theta\phi(\bar{w}) - C\bar{x})\bar{w}.$$

The proof of Theorem 7.3 begins with establishing that, for every function $h : \mathbf{R} \mapsto \mathbf{R}$ with L1 norm not exceeding 1, and for every square integrable function $w : \mathbf{R} \mapsto \mathbf{R}$ the integral

$$\int_{-\infty}^\infty (w(t) - \phi(w(t)))y(t) dt,$$

where $y = h * w$, is non-negative. This verifies that the assumptions of Theorem 7.2 are satisfied, and proves existence of the corresponding storage function without actually finding it. Combining the Zames-Falb supply rate with the statement of the Kalman-Yakubovich-Popov lemma yields the following stability criterion.

Theorem 7.4 *Assume that matrices* A_p, B_p, C_p *are such that* A_p *is a Hurwitz matrix, and there exists* $\epsilon > 0$ *such that*

$$\text{Re}(1 - G(j\omega))(1 - H(j\omega)) \geq \epsilon \quad \forall \; \omega \in \mathbf{R},$$

where H *is a Fourier transform of a function with L1 norm not exceeding 1, and*

$$G(s) = C_p(sI - A_p)^{-1}B_p.$$

Then system

$$\dot{x}(t) = A_p x(t) + B_p \phi(Cx(t) + v(t))$$

has finite L2 gain, in the sense that there exists $\gamma > 0$ *such that*

$$\int_0^\infty |x(t)|^2 dt \leq \gamma(|x(0)|^2 + \int_0^\infty |v(t)|^2 dt$$

for all solutions.

7.4   Example with cubic nonlinearity and delay

Consider the following system of differential equations[2] with an uncertain constant delay parameter τ:

$$
\begin{aligned}
\dot{x}_1(t) &= -x_1(t)^3 - x_2(t-\tau)^3 & (7.16)\\
\dot{x}_2(t) &= x_1(t) - x_2(t) & (7.17)
\end{aligned}
$$

Analysis of this system is easy when $\tau = 0$, and becomes more difficult when τ is an arbitrary constant in the interval $[0, \tau_0]$. The system is not exponentially stable for any value of τ. Our objective is to show that, despite the absence of exponential stability, the method of storage functions with quadratic supply rates works.

The case $\tau = 0$

For $\tau = 0$, we begin with describing (7.16),(7.17) by the behavior set

$$\mathcal{Z} = \{z = [x_1; x_2; w_1; w_2]\},$$

where

$$w_1 = x_1^3, \; w_2 = x_2^3, \; \dot{x}_1 = -w_1 - w_2, \; \dot{x}_2 = x_1 - x_2.$$

Quadratic supply rates for which follow from the *linear* equations of $\mathcal{Z}$ are given by

$$\sigma_{LTI}(z) = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' P \begin{bmatrix} -w_1 - w_2 \\ x_1 - x_2 \end{bmatrix},$$

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[2]Suggested by Petar Kokotovich

where $P = P'$ is an arbitrary symmetric 2-by-2 matrix defining storage function

$$V_{LTI}(z(\cdot), t) = x(t)'Px(t).$$

Among the non-trivial quadratic supply rates σ valid for $\mathcal{Z}$, the simplest are defined by

$$\sigma_{NL}(z) = d_1 x_1 w_1 + d_2 x_2 w_2 + q_1 w_1 (-w_1 - w_2) + q_2 w_2 (x_1 - x_2),$$

with the storage function

$$V_{NL}(z(\cdot), t) = 0.25(q_1 x_1(t)^4 + q_2 x_2(t)^4),$$

where $d_k \geq 0$. It turns out (and is easy to verify) that the only convex combinations of these supply rates which yield $\sigma \leq 0$ are the ones that make $\sigma = \sigma_{LTI} + \sigma_{NL} = 0$, for example

$$P = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \qquad d_1 = d_2 = q_2 = 1, \qquad q_1 = 0.$$

The absence of strictly negative definite supply rates corresponds to the fact that the system is not exponentially stable. Nevertheless, a Lyapunov function candidate can be constructed from the given solution:

$$V(x) = x'Px + 0.25(q_1 x_1^4 + q_2 x_2^4) = 0.5x_1^2 + 0.25x_2^4.$$

This Lyapunov function can be used along the standard lines to prove global asymptotic stability of the equilibrium $x = 0$ in system (7.16),(7.17).

7.4.1 The general case

Now consider the case when $\tau \in [0, 0.2]$ is an uncertain parameter. To show that the delayed system (7.16),(7.17) remains stable when $\tau \leq 0.2$, (7.16),(7.17) can be represented by a more elaborate behavior set $\mathcal{Z} = \{z(\cdot)\}$ with

$$z = [x_1; x_2; w_1; w_2; w_3; w_4; w_5; w_6] \in \mathbf{R}^8,$$

satisfying LTI relations

$$\dot{x}_1 = -w_1 - w_2 + w_3, \quad \dot{x}_2 = x_1 - x_2$$

and the nonlinear/infinite dimensional relations

$$w_1(t) = x_1^3, \ w_2 = x_2^3, \ w_3 = x_2^3 - (x_2 + w_4)^3,$$

$$w_4(t) = x_2(t - \tau) - x_2(t), \ w_5 = w_4^3, \ w_6 = (x_1 - x_2)^3.$$

Some additional supply rates/storage functions are needed to bound the new variables. These will be selected using the perspective of a small gain argument. Note that the

perturbation w_4 can easily be bounded in terms of $\dot{x}_2 = x_1 - x_2$. In fact, the LTI system with transfer function $(\exp(-\tau s) - 1)/s$ has a small gain (in almost any sense) when τ is small. Hence a small gain argument would be applicable provided that the gain "from w_4 to $\dot{x}_2$" could be bounded as well.

It turns out that the L_2-induced gain from w_4 to $\dot{x}_2$ is unbounded. Instead, we can use the L_4 norms. Indeed, the last two components w_5, w_6 of w were introduced in order to handle L_4 norms within the framework of quadratic supply rates. More specifically, in addition to the usual supply rate

$$\sigma_{LTI}(z) = 2 \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]' P \left[\begin{array}{c} -w_1 - w_2 + w_3 \\ x_1 - x_2 \end{array} \right],$$

the set $\mathscr{Z}$ has supply rates

$$\begin{aligned} \sigma(z) =& d_1 x_1 w_1 + d_2 x_2 w_2 + q_1 w_1(-w_1 - w_2 + w_3) + q_2 w_2(x_1 - x_2) \\ &+ d_3[0.99(x_1 w_1 + x_2 w_2) - x_1 w_3 + 2.5^4 w_4 w_5 - 0.5^4(x_1 - x_2)w_6] \\ &+ q_3[0.2^4(x_1 - x_2)w_6 - w_4 w_5], \end{aligned}$$

$d_i \geq 0$. Here the supply rates with coefficients d_1, d_2, q_1, q_2 are same as before. The term with d_3, based on a zero storage function, follows from the inequality

$$0.99(x_1^4 + x_2^4) - x_1(x_2^3 - (x_2 + w_4)^3) + \left(\frac{5w_4}{2} \right)^4 - \left(\frac{x_1 - x_2}{2} \right)^4 \geq 0$$

(which is satisfied for *all* real numbers x_1, x_2, w_4, and can be checked numerically).

The term with q_3 follows from a gain bound on the transfer function $G_\tau(s) = (\exp(-\tau s) - 1)/s$ from $x_1 - x_2$ to w_4. It is easy to verify that the L_1 norm of its impulse response equals τ, and hence the L^4 induced gain of the causal LTI system with transfer function G_τ will not exceed 1. Consider the function

$$V_d(v(\cdot), T) = -\inf \int_T^\infty \left\{ 0.2^4 |v_1(t)|^4 - \left| \int_{t-\tau}^t v_1(r)dr \right|^4 \right\} dt, \tag{7.18}$$

where the infimum is taken over all functions v_1 which are square integrable on $(0, \infty)$ and such that $v_1(t) = v(t)$ for $t \leq T$. Because of the L_4 gain bound of G_τ with $\tau \in [0, 0.2]$ does not exceed 0.2, the infimum in (7.18) is bounded. Since we can always use $v_1(t) = 0$ for $t > T$, the infimum is non-positive, and hence V_d is non-negative. The IQC defined by the "q_3" term holds with $V_\sigma = q_3 V_d(x_1 - x_2, t)$.

Let

$$\sigma_0(z) = -0.01(x_1 w_1 + x_2 w_2) = -0.01(x_1^4 + x_2^4),$$

which reflects our intention to show that x_1, x_2 will be integrable with fourth power over $(0, \infty)$. Using

$$P = \left[\begin{array}{cc} 0.5 & 0 \\ 0 & 0 \end{array} \right], \quad d_1 = d_2 = 0.01, \ d_3 = q_2 = 1, \ q_1 = 0, \ q_3 = 2.5^4$$

yields a Lyapunov function

$$V(x_e(t)) = 0.5x_1(t)^2 + 0.25x_2(t)^4 + 2.5^4 V_d(x_1 - x_2, t),$$

where x_e is the "total state" of the system (in this case, $x_e(T) = [x(T); v_T(\cdot)]$, where $v_T(\cdot) \in L_2(0, \tau)$ denotes the signal $v(t) = x_1(T - \tau + t) - x_2(T - \tau + t)$ restricted to the interval $t \in (0, \tau)$). It follows that

$$\frac{dV(x_e(t))}{dt} \leq -0.01(x_1(t)^4 + x_2(t)^4).$$

On the other hand, we saw previously that $V(x_e(t)) \geq 0$ is bounded from below. There-fore, $x_1(\cdot), x_2(\cdot) \in \Lambda_4$ (fourth powers of x_1, x_2 are integrable over $(0, \infty)$) as long as the initial conditions are bounded. Thus, the equilibrium $x = 0$ in system (7.16),(7.17) is stable for $0 \leq \tau \leq 0.2$.