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6.245: MULTIVARIABLE CONTROL SYSTEMS

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Solving the H2 optimization problem[1]

There are several ways to derive a solution to the H2 optimization problem. The path developed below relies on reduction of the output feedback design problem to an "abstract" optimal program control problem.

3.1 A Characterization of Closed Loop Impulse Responses

This section provides an insight into the constraints imposed on the closed loop system by the coefficients of the ouput feedback design plant

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, && (3.1) \\ y &= C_2 x + D_{21} w, && (3.2) \end{aligned}$$

stabilized by a finite order strictly proper LTI controller

$$\begin{aligned} \dot{x}_f &= A_f x_f + B_f y, && (3.3) \\ u &= C_f x_c. && (3.4) \end{aligned}$$

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3.1.1 An Affine Parameterization

Let $X = X(t)$ denote the impulse response matrix from w to x in the closed loop system. Similarly, let $U = U(t)$ be the impulse response matrix from w to u.

Theorem 3.1 *Assume (A, B_2) is stabilizable and (C_2, A) is detectable. Then a pair $(X(t), U(t))$ of matrix-valued functions can be achieved as implulse responses in a stable closed loop system if and only if*

$$\dot{X}(t) = AX(t) + B_2U(t),\ \ X(0) = B_1,\ \lim_{t\to\infty} X(t) = 0, \tag{3.5}$$

and there exist a real matrix-valued function $V = V(t)$, each entry of which is a linear combination of terms of the form $t^k e^{-st}$ with $k \in \{0, 1, \dots\}$, $\mathrm{Re}(s) > 0$, such that

$$U(t) = \Psi(t)B_1 + V(t)D_{21},\ \lim_{t\to\infty} \Psi(t) = 0, \tag{3.6}$$

where

$$\dot{\Psi}(t) = \Psi(t)A + V(t)C_2,\ \ \Psi(0) = 0. \tag{3.7}$$

Equality (3.5) represents the constraints on the closed loop responses in the case of "full information control", whem x and w are available for measurement. When output feedback is considered, $U(t)$ is not arbitrary, but is parameterized by (3.6),(3.7).

Theorem 3.1 can be generalized to the case of finite H2 norm stabilization by infinite order LTI controllers, in which case the conditions requiring $X(t), \Psi(t)$ and $\Psi(t)$ to converge to zero as $t \to \infty$ should be replaced by the condition of their square integrability over $\{t\} = (0, \infty)$.

Proof For the purpose of driving the H2 optimal controller, we do not need the sufficiency part of the theorem. It will be proven later, together with a slight modification of Theorem 3.1 concerning the so-called "Q-parameterization".

It is sufficient to prove the theorem for the case when

$$B_1 = \left[\begin{array}{cc} I & 0 \end{array}\right],\ \ D_{21} = \left[\begin{array}{cc} 0 & I \end{array}\right],$$

i.e. when system equations have the form

$$\dot{x} = Ax + B_2u + w_1, \tag{3.8}$$

$$y = C_2x + w_2 \tag{3.9}$$

with $w = [w_1; w_2]$. Then (3.5) follows from (3.8). One way to see this is by applying the one-sided Laplace transform (denoted by tildes) to (3.8) to get

$$s\tilde{x} = A\tilde{x} + B_2\tilde{u} + \tilde{w}_1.$$

Since

$$\tilde{x} = \tilde{X}\tilde{w},\ \tilde{u} = \tilde{U}\tilde{w},\ \tilde{w}_1 = \left[\begin{array}{cc} I & 0 \end{array}\right]\tilde{w},$$

and $\tilde{w}$ is arbitrary, (3.5) follows.

To prove (3.6) and (3.7), note that, by linearity of the controller, $u(t) \equiv 0$ whenever $y(t) \equiv 0$. Hence

$$\tilde{w}_2 = -C_2(sI - A)^{-1}\tilde{w}_1$$

would always imply

$$\tilde{u} = \tilde{U}\left[\begin{array}{c} I \\ -C_2(sI - A)^{-1} \end{array}\right]\tilde{w}_1 = 0$$

for an arbitrary $\tilde{w}_1$. Therefore

$$\tilde{U} = \left[\begin{array}{cc} \tilde{V}C_2(sI - A)^{-1} & \tilde{V} \end{array}\right]$$

for some V. Converting this to time domain terms yields $U = [\Psi \ \ V]$ where

$$\dot{\Psi}(t) = \Psi(t)A + V(t)C_2, \quad \Psi(0) = 0.$$

■

3.1.2 A Characterization of State Estimation Errors

As a special case of the general closed loop response parameterization, consider the closed loop response from w to $Kx - u$, where K is a given matrix.

Theorem 3.2 *A matrix function $G = G(t)$ can be achieved as a closed loop impulse response from w to $Kx - u$ if and only if there exists a real matrix-valued function $V = V(t)$, each entry of which is a linear combination of terms of the form $t^k e^{-st}$ with $k \in \{0, 1, \dots\}$, $\mathrm{Re}(s) > 0$, such that*

$$G(t) = \Psi(t)B_1 + V(t)D_{21},\ \lim_{t\to\infty}\Psi(t) = 0, \tag{3.10}$$

where

$$\dot{\Psi}(t) = \Psi(t)A + V(t)C_2,\ \Psi(0) = K. \tag{3.11}$$

Proof For now, we will only prove necessity (this is the part we need for H2 optimization). According to Theorem 3.1, a stabilized closed loop impulse response from w to $Kx - u$ equals the response H of system

$$\begin{aligned} \dot{X} &= AX + B_2(\Psi B_1 + VD_{21}), \quad X(0) = B_1, && (3.12)\\ \dot{\Psi} &= \Psi A + VC_2, \quad \Psi(0) = 0, && (3.13)\\ H &= KX - \Psi B_1 - VD_{21} && (3.14) \end{aligned}$$

to an input V which makes $X(\infty) = 0$ and $\Psi(\infty) = 0$.

Note that the zero input response of system (3.12), (3.13) (i.e. with $V \equiv 0$) has $\Psi \equiv 0$ and hence $H(t) = H_0(t) = Ke^{At}B_1$. This response can be achieved in (3.10),(3.11) with $V \equiv 0$. On the other hand, according to Theorem 3.1, every zero-state response of the system (i.e. when $X(0) = B_1$ is replaced by $X(0) = 0$) must coincide with a zero-state response of (3.10),(3.11). ■

3.2 Abstract H2 Optimization

We will use the term "abstract H2 optimization" to refer to an auxiliary optimization problem: an optimal program control in completely deterministic settings. It turns out that a general H2 feedback design problem reduces to a *pair* of abstract H2 optimizations, and the optimal controller, as well as a parameterization of suboptimal controllers, can be obtained easily from solutions of the two auxiliary problems.

3.2.1 Formal Definitions and Relation to Output Feedback Design

An *abstract H2 optimization problem* is defined by a stabilizable LTI system

$$\mathcal{S} = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array} \right).$$

and requests finding, for every fixed vector p_0 in the state space, a function of time $q = q(t)$ (bounded for $t \in [0, \infty)$ and converging to zero as $t \to \infty$) such that the solution $p = p(t)$ of

$$\dot{p}(t) = ap(t) + bq(t), \quad p(0) = p_0 \tag{3.15}$$

satisfies

$$\lim_{t \to \infty} p(t) = 0, \tag{3.16}$$

and, subject to this constraints, minimizes the quadratic integral

$$\Phi(u(\cdot)) = \int_0^\infty |cp(t) + dq(t)|^2 dt \to \min. \tag{3.17}$$

One motivation for considering abstract H2 optimization is the "full information" version of the standard H2 output feedback design. According to Theorem 3.1, the closed loop H2 norm can be expressed as the integral

$$\Phi_{fi} = \int_0^\infty \|C_1X(t) + D_{12}U(t)\|_F^2 dt,$$

where

$$\|M\|_F^2 = \mathrm{trace}(M'M)$$

denotes the square of the Frobenius norm of matrix M, and X, U are the closed loop impulse responses from w to x and u respectively, constrained only by

$$\dot{X} = AX + B_2U, \ X(0) = B_1, \ X(\infty) = 0, \ U(\infty) = 0.$$

If X_i, U_i, B_{1i} denote the i-th column of X, U, and B_1 respectively, Φ_{fi} can be written as

$$\Phi_{fi} = \sum_i \int_0^\infty |C_1X_i(t) + D_{12}U_i(t)|^2 dt.$$

Since every 3-typle $(X_i, U_i, B_{1i}$ is independently constrained by

$$\dot{X}_i(t) = AX_i(t) + B_2U_i(t), \ X_i(0) = B_{1i},$$

the task of minimizing Φ_{fi} decomposes into independent abstract H2 minimizations with

$$a = A, \ b = B_2, \ c = C_1, \ d = D_{12}, \ p_0 = B_{1i}.$$

A similar motivation for abstract H2 optimization comes from an attempt to minimize the closed loop H2 norm from w to $Kx - u$, where K is a given matrix. According to Theorem 3.2, the closed loop impulse response can be chosen according to

$$H = \Psi B_1 + VD_{21}, \quad \dot{\Psi} = \Psi A + VC_2, \ \Psi(0) = K.$$

Let Ψ_i, V_i, K_i be the i-th *row* of Ψ, V, K respectively. The H2 norm of H equals the sum of the integrals

$$\int_0^\infty |\Psi_i(t)B_1 + V_i(t)D_{21}|^2 dt,$$

minimizing which leads to solving a family of independent abstract H2 optimization problems with

$$a = A', \ b = C_2', \ c = B_1', \ d = D_{21}', \ p_0 = K_i'.$$

3.2.2 The "easy" version of the KYP Lemma

The so-called "Kalman-Yakubovich-Popov Lemma" (also frequently referred to as the "Positive Real Lemma" provides, among other things, a complete solution to abstract H2 optimization. The much simplified version of the KYP Lemma we need here will be complemented by a full statement used extensively in H-Infinity optimization and optimization relying on semidefinite programming.

Theorem 3.3 *Assume that pair (a, b) is stabilizable. Then the following conditions are equivalent:*

(a) a unique optimal control exists for every p_0 in the abstract H2 optimization problem (3.15)-(3.17);

(b) matrix

$$M(j\omega) = \begin{bmatrix} a - j\omega I & b \\ c & d \end{bmatrix}$$

is left invertible for all $\omega \in \mathbf{R}$ and at $\omega = \infty$ (which means that d is left invertible as well);

(c) $d'd > 0$, and matrix

$$\mathcal{H} = \begin{bmatrix} a - b(d'd)^{-1}d'c & b(d'd)^{-1}b' \\ c'c - c'd(d'd)^{-1}d'c & -a' + c'd(d'd)^{-1}b' \end{bmatrix}$$

has no eigenvalues on the imaginary axis;

(d) $d'd > 0$ and there exist (unique) matrices $\kappa = \kappa' \geq 0$ and k such that $a + bk$ is a Hurwitz matrix, and

$$|cp + dq|^2 + 2p'\kappa(ap + bq) = (q - kp)'d'd(q - kp) \tag{3.18}$$

for all vectors p, q.

If conditions (a)-(d) are satisfied then the optimal q is defined by the relation $q = kp$, i.e.

$$q_*(t) = ke^{(a+bk)t}p_0,$$

and the minimal cost equals $p_0'\kappa p_0$. Moreover, the dimension of the strictly stable invariant subspace $\mathcal{V}_+$ of $\mathcal{H}$ will equal the number n of components of p, and κ can be found from

$$\kappa = -\begin{bmatrix} \psi_1 & \psi_2 & \dots & \psi_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^{-1},$$

where

$$\left\{ \begin{bmatrix} x_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} x_n \\ \psi_n \end{bmatrix} \right\}$$

is an arbitrary basis in $\mathcal{V}_+$.

Matrix $\mathcal{H}$ is usually called the *Hamiltonian matrix* associated with the abstract H2 optimization. Condition (c) provides a convenient way of checking the non-singularity imposed by (b). Condition (d) defines a *completion of squares* procedure, which is a natural and intuitive way of solving the abstract H2 optimization problem. By comparing the coefficients on both sides of (3.18), one can derive a quadratic algebraic equation for κ, called the *algebraic Riccati equation* (ARE):

$$\alpha + \kappa\beta + \beta'\kappa = \kappa\gamma\kappa,$$

where α, β, γ are the coefficients of the Hamiltinian matrix:

$$\mathcal{H} = \begin{bmatrix} \beta & \gamma \\ \alpha & -\beta' \end{bmatrix}.$$

A solution κ for which

$$a + bk = \beta - \gamma\kappa$$

is a Hurwitz matrix is called a *stabilizing solution* of the ARE. A stabilizing solution, if it exists, is unique. It defines the "optimal controller" gain k in $q = kp$, though, formally, abstract H2 optimization is about program control optimization.

We will postpone proving the KYP Lemma until its full version is formulated.

3.2.3 Solution of Feedback H2 Optimization

In order to solve the standard H2 feedback optimization problem, consider the two auxiliary abstract H2 optimization setups, the "full information control" version, defined with

$$a = A,\ b = B_2,\ c = C_1,\ d = D_{12}, \tag{3.19}$$

and the "state estimation" version, defined by

$$a = A',\ b = C_2',\ c = B_1',\ d = D_{21}'. \tag{3.20}$$

It is easy to verify by inspection that absense of control singularity in the original feedback design setup is equivalent to satisfying condition (b) of Theorem 3.3 in the abstract setup defined by (3.19), and absense of sensor singularity is equivalent to satisfying (b) in the abstract setup defined by (3.19). Hence, in the non-singular case, the corresponding completion of squares is possible, with matrices $\kappa = P_{fi}$, $k = K_{fi}$ in the control setup, $\kappa = P_{se}$ and $k = K_{se}$ in the state estimation setup.

Theorem 3.4 *The H2 optimal controller is defined by*

$$\begin{aligned} u(t) &= K_{fi}\hat{x}(t), && (3.21)\\ \frac{d\hat{x}(t)}{dt} &= A\hat{x}(t) + B_2u(t) + K'_{se}(C_2\hat{x}(t) - y(t)), && (3.22) \end{aligned}$$

and provides the minimal H2 norm (squared) of

$$J_* = \mathrm{trace}(B_1'P_{fi}B_1 + \mathrm{trace}(D_{12}K_{fi}P_{se}K'_{fe}D'_{12}).$$

Proof The overall closed loop H2 norm (squared) is given by

$$J = \int_0^\infty \|C_1X(t) + D_{12}U(t)\|_F^2 dt.$$

According to the definition of K_{fi} and P_{fi}, this integral equals

$$J = \mathrm{trace}(B_1'P_{fi}B_1 + \int_0^\infty \|D_{12}(U - K_{fi}X)\|_F^2 dt.$$

This can be interpreted as saying that, in the H2 optimal closed loop system, u must be the best estimate of $K_{fi}x$, in the sense that the H2 norm from w to $D_{12}(u - K_{fi}x)$ should be minimal. According to the definition of K_{se} and P_{se}, the minimal estimation error equals

$$\mathrm{trace}(D_{12}K_{fi}P_{se}K'_{fe}D'_{12}),$$

and is achieved when the impulse response G from w to $K_{fi}x - u$ is given by

$$G(t) = \Psi(t)B_1 + V(t)D_{21},$$

where

$$\dot{\Psi}(t) = \Psi(t)A + V(t)C_2,\ \Psi(0) = K,$$

and

$$V(t) = \Psi(t)K_{se}.$$

By inspection, this optimal impulse response is achieved by the controller in the formulation of the theorem. ■