Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS

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Fundamentals of Model Order Reduction[1]

This lecture introduces basic principles of model order reduction for LTI systems, which is about finding good low order approximations of high order systems.

8.1 Setting Up A Model Order Reduction Problem

This section describes a number of different ways of setting up a model reduction problem. In contrast with the case of LTI feedback optimization, when the most natural problem formulation has a satisfactory solution (H2 or H-Infinity optimization via Riccati equations), no efficient solution is known for the most natural formulation of the model order reduction problem. This leads to a variety of compromise approaches.

8.1.1 An abstract setup

To define a dynamical system model complexity reduction problem, one typically needs (at least) the following two ingredients.

(a) A numerical measure of complexity, applicable to system models under consideration. For example, complexity of an LTI system can be measured by its order, while complexity of a finite state automata can be measured by the number of its states.

(b) A numerical measure of distance between two systems. For example, the L2 norm of the difference of impulse responses can serve as a measure of distance between two LTI systems.

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A typical model reduction problem will then be formulated as the task of finding a "reduced" system $\hat{S} = \hat{S}_k$ of complexity not larger than a given threshold k, such that the distance between $\hat{S}$ and a given "complex" system S is as small as possible. Alternatively, a maximal admissible distance between S and $\hat{S}$ can be given, in which case the complexity k of $\hat{S}$ is to be minimized.

As is suggested by the experience of robust control, useful measures of distance between S and $\hat{S}$ can be defined as the induced gain (for example, the L2 gain) from a testing input f to the output matching error e, as shown on Figure 8.1. In this case, for design

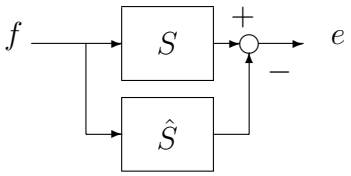

Figure 8.1: Comparing S and $\hat{S}$

or analysis purposes, S can be represented as a series connection (i.e. a sum) of $\hat{S}$ and an "uncertain" error system $\Delta = S - \hat{S}$.

8.1.2 H-Infinity Optimal Model Reduction Setup

For LTI systems G given in an input/output format (for example, by their transfer matrices $G = G(s)$ or by the impulse response $g = g(t)$), complexity is well represented by the *system order*, defined as the number of states in a state-space realization which is both controllable and observable (also called the McMillan degree).

The H-Infinity model reduction problem is that of finding a stable LTI transfer matrix $\hat{G} = \hat{G}_k$ of order smaller than a given number k such that $\|W^{-1}(G - \hat{G}_k)\|_\infty$ is as small as possible, where G, W are given stable transfer matrices (W^{-1} is also assumed to be stable), and $\|\Delta\|_\infty$ denotes H-Infinity norm (L2 gain) of a stable system Δ. As a result of model order reduction, G can be represented as a series connection of a lower order "nominal plant" $\hat{G}$ and a bounded uncertainty (see Figure 8.2). In most cases, (even when $W \equiv 1$) the H-Infinity optimal model order reduction is a problem without a known good solution. This can be explained (informally) as a consequence of non-convexity of the set of all stable transfer matrices of order less than k. In particular, one can expect many local minima in an attempt at a gradient-based optimization of $\|W^{-1}(G - \hat{G})\|_\infty$.

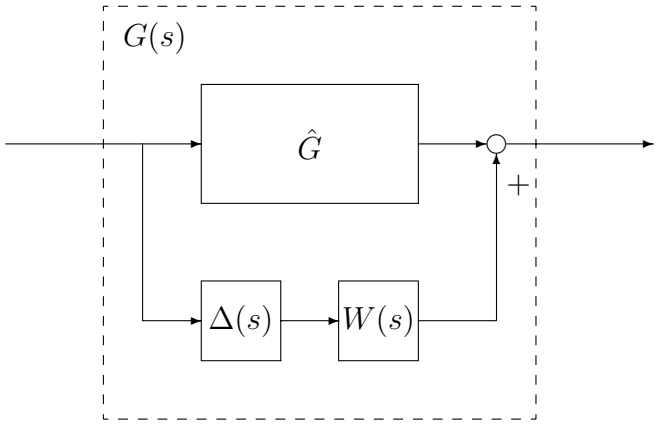

Figure 8.2: Model order reduction and uncertain models

8.1.3 Approximation by Linear Combinations vs. Model Reduction

A much easier problem is that of finding the best approximation of a given LTI system G by a linear combination $\hat{G}$ of *given* LTI systems G_k:

$$\|G - \hat{G}\| \to \min, \quad \hat{G} = \sum c_r G_r,$$

where $\|\cdot\|$ is some system norm. For many approximation quality measures, this problem has a relatively simple solution. For example, if $\|\cdot\| = \|\cdot\|_2$ is the H2 norm then the optimization becomes a standard least squares problem reducible to solving a system of linear equations. If $\|\cdot\| = \|\cdot\|_\infty$ is the H-infinity norm, the optimization is reduced to solving a system of Linear Matrix Inequalities (LMI), a special class of convex optimization problems solvable by an efficient algorithm, to be discussed later in the course.

While the technique of approximation by linear combinations should not be expected to yield a close-to-optimal relation between the order of $\hat{G}$ and the size of model matching error $\|G - \hat{G}\|$, it is frequently used in the initial stage of a model reduction process as a tool for approximating infinite dimentional systems by finite order models. In particular, a least squares algorithm can be used to derive a high quality approximation G_0 of a non-rational transfer matrix G, such that G_0 has a high but finite order. Then, if $\|G - G_0\|_\infty$ is small enough, model reduction of G can be replaced by model reduction of G_0.

8.1.4 Hankel Model Reduction

The so-called *Hankel optimal model reduction* algorithm is based on finding a stable m-th order LTI approximation $\hat{G}$ of a given stable LTI system G which minimizes the so-

called *Hankel norm* $\|G - \hat{G}\|_H$ of model reduction error. Technical details of definitions and optimization algorithms associated with the Hankel norm will be discussed in the following sections. For the user, it is important to know that Hankel norm $\|\Delta\|_H$ of a stable system Δ is never larger than its H-Infinity norm $\|\Delta\|_\infty$, hence solving the Hankel norm optimal model reduction problem yields a *lower bound* for the minimum in the H-Infinity norm optimal model reduction. Moreover, H-Infinity norm of model reduction error associated with a Hankel norm optimal reduced model is typically close to this lower bound. Thus, Hankel norm optimal reduced model can serve well as an H-Infinity suboptimal reduced model.

Hankel optimal model reduction appears to be one of the best model reduction options available. However, this approach also has some weak spots. One problem is the cubic (with respect to the order of the original system G) growth of complexity of the model reduction algorithm, which makes it impractical to apply it to systems of order 10^3 and larger. Another (perhaps related) weakness lies in the difficulty of adapting the algorithm to a situation when G is not perfectly known, and, alternatively, samples of its time or frequency response are available.

8.1.5 Non-Optimizing Model Reduction

Due to the prohibitively high cost of Hankel model reduction, applications dealing with very large order systems frequently resort to numerically inexpensive methods not promising any optimality at all. We will refer to them as *non-optimizing* model reduction tools, stressing that no model reduction error is minimized when the methods are applied. In this subsection we introduce some of these methods briefly, using the SISO model reduction, in which a given stable transfer function $G = G(s)$ is to be approximated by the ratio $\hat{G}(s) = p(s)/q(s)$, where p, q are polynomials of order m.

One popular approach is *moments matching*. In the simplest form of moments matching, an m-th order approximation $\hat{G}(s) = p(s)/q(s)$ (where p, q are polynomials of order m) of a SISO transfer function $G(s)$ is defined by matching the first $2m + 1$ coefficients of a Taylor expansion

$$G(s) = G(0) + G'(0)s + \frac{1}{2}G''(0)s^2 + \dots$$

by the coefficients of a similar expansion of $\hat{G}$. This way, one gets a system of $2m + 1$ equations with $2m + 1$ parameters. There is a whole branch of model reduction science of writing these equations in a numerically robust form, of using other (or multiple) frequencies for matching Taylor expansion coefficients, etc. However, the following major flaw remains with the approach: it does not guarantee any degree of near-optimality of the

reduced models, nor even *stability* of the resulting $\hat{G}$. In practice, the moments matching approach looses dramatically to Hankel and other optimization-based model reduction algorithms.

Another popular method of model reduction is based on minimizing the error of matching a fixed output. For example, assume that samples $y_k = y(kT)$, $k = 1, 2, \dots$ at a fixed sampling rate $T > 0$ are available for the impulse response of a given CT LTI system. If the system has order m, there would exist a Schur polynomial

$$q(z) = q_m z^m + q_{m-1} z^{m-1} + \cdots + q_1 z + q_0,$$

with $q_m \neq 0$, such that

$$q_m y_{k+m} + q_{m-1} y_{k+m-1} + \cdots + q_1 y_{k+1} + q_0 y_k = 0$$

for all $k > 0$. The idea is to define the denominator of the reduced model $\hat{G}(s)$ in terms of a polynomial $q = q(z)$ which minimizes the sum of squares of

$$e_k = q_m y_{k+m} + q_{m-1} y_{k+m-1} + \cdots + q_1 y_{k+1} + q_0 y_k$$

subject to a normalization constraint imposed on the coefficients of q (otherwise the minimum is achieved at $q \equiv 0$). When the normalization is given by $q_m = 1$, we get a classical least squares system identification algorithm. When the normalization is given by

$$\sum_{k=0}^{m} |q_k|^2 = 1,$$

we get a version of the *proper orthogonal decomposition* algorithm. The trouble with these approaches is very much the same as with moments matching: no reason to expect near-optimality of the reduced model, and, unless significant performance-reducing modifications are introduced, the reduced model generated by these algorithms is not necessarily stable.

8.2 Hankel Operators and System Balancing

Here we introduce the fundamental notions of Hankel operator and Hankel singular numbers associated with a given stable LTI system. For practical calculations, balancing of stable LTI system is defined and explained.

8.2.1 Hankel Operator

The "convolution operator" $f \mapsto y = g * f$ associated with a LTI system with impulse response $g = g(t)$ has infinite rank whenever g is not identically equal to zero. However, with every LTI system of finite order, it is possible to associate a meaningful and representative linear transformation of finite rank. This transformation is called the *Hankel operator*.

A Hankel operator $\mathcal{H}_G$ can be associated with every LTI system G which has a finite L2 gain. It is important that G does not have to be causal. The domain $\mathcal{D}(\mathcal{H}_G)$ of $\mathcal{H}_G$ consists of all fignals $f = f(t)$ of finite energy such that $f(t) = 0$ for $t \geq 0$. The result of applying the Hankel operator $\mathcal{H}_G$ to a signal $f \in \mathcal{D}(\mathcal{H}_G)$ is a signal $y = y(t)$ which, for $t > 0$, equals the response of G to input f, and equals zero otherwise. In other terms, $\mathcal{H}_G :\ L_2(-\infty, 0) \mapsto L_2(0, \infty)$ is defined by

$$(\mathcal{H}_G f)(t) = \int_0^\infty g(t+\tau) f(-\tau) d\tau,$$

where $g = g(t)$ is the impulse response of G.

Let $G = G(s)$ be a m-by-k matrix-valued function of complex argument which is defined on the extended imaginary axis $j\mathbf{R} \cup \{\infty\}$, where it satisfies the conditions of real symmetry (i.e. elements of $G(-j\omega)$ are the complex conjugates of the corresponding entries of $G(j\omega)$) and continuity (i.e. $G(j\omega)$ converges to $G(j\omega_0)$ as $\omega \to \omega_0$ for all $\omega_0 \in \mathbf{R}$ and for $\omega_0 = \infty$). Note that a rational function $G = G(s)$ with real coefficients satisfies this condition if and only if it is proper and has no poles on the imaginary axis.

Let $L_2^k(-\infty, 0)$ denote the set of square integrable functions $f :\ \mathbf{R} \mapsto \mathbf{R}^k$ such that $f(t) = 0$ for $t \geq 0$. Let $L_2^m(0, \infty)$ be the set of square integrable functions $g :\ \mathbf{R} \mapsto \mathbf{R}^m$ such that $g(t) = 0$ for $t \leq 0$. The *Hankel operator* associated with G is a map $\mathcal{H}_G :\ L_2^k(-\infty, 0) \mapsto L_2^m(0, \infty)$ defined by

$$(\mathcal{H}_G f)(t) = g(t) = \begin{cases} g_0(t), & t > 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $g_0 = g_0(t)$ is the inverse Fourier transform of $\tilde{g}_0(j\omega) = G(j\omega)\tilde{f}(j\omega)$. Less formally, the Hankel operator $\mathcal{H}_G$ associated with an LTI system G takes an input f which equals zero for for $t \geq 0$, and produces a signal g which equals the corresponding output of G for $t > 0$, and equals zero for $t \leq 0$.

8.2.2 Rank and Gain of a Hankel Operator

As a linear transformation $\mathcal{H}_G :\ L_2^k(-\infty, 0) \mapsto L_2^m(0, \infty)$, a Hankel operator has its rank and L2 gain readily defined: the *rank* of $\mathcal{H}_G$ is the maximal number of linearly independent

outputs (could be plus infinity), the *L2 gain* is the minimal upper bound for the L2 norm of $\mathcal{H}_G f$ subject to the L2 norm of f being fixed at one.

Remember that the order of a rational transfer matrix G is defined as the order of its controllable and observable realization. The order of a non-rational transfer matrix equals infinity. The *L-Infinity* norm of a transfer matrix is defined similarly to the H-Infinity norm, but without the requirement of stability:

$$\|G\|_\infty = \sup_{\omega \in \mathbf{R}} \sigma_{\max}(G(j\omega)).$$

There is a simple but very useful relation between the rank and norm of a Hankel operator $\mathcal{H}_G$, on one hand, and the order and H-Infinity norm of the original LTI system G, on the other.

Theorem 8.1 *For all transfer matrices G, the rank of $\mathcal{H}_G$ does not exceed the order of G, and the L2 gain of $\mathcal{G}$ does not exceed the L-Infinity norm of G. Moreover, for* stable *transfer matrices G, the rank of $\mathcal{H}_G$* equals *the order of G.*

In other words, rank of $\mathcal{H}_G$ equals the order of the stable part of G, and L2 gain of $\mathcal{H}_G$ is never larger than $\|G\|_\infty$.

Proof To show the gain/L-Infinity norm relation, let us return to the definition of $\mathcal{H}_G$ in the previous subsection. Note that the L2 norm of g_0 is not larger than $\|G\|_\infty \|f\|_2$. On the other hand, $\|g\|_2 \leq \|g\|_2$.

To show that the rank of $\mathcal{H}_G$ equals the order of the stable part of G, note first that the *unstable* part of G does not have any effect of $\mathcal{H}_G$ at all. Then, for a stable G, the map of f into the inverse Fourier transform g_0 of $\tilde{g}_0 = G\tilde{f}$ is a causal linear system. Hence, the only way for a past signal $f \in L_2(-\infty, 0)$ to affect the future of the output is through the system's state at zero. ■

Naturally, the so-called *Hankel norm* $\|G\|_H$ of G is defined as the L2 gain of the corresponding Hankel operator $\mathcal{H}_G$.

8.2.3 Matrix rank reduction

The common feature matrices and linear systems share is that both describe *linear transformations*. Before addressing the problem of LTI system model order reduction, it is instructive to consider the case of *matrices*, in which a similar task is defined as the *matrix rank reduction problem*: given a matrix M, and an integer $k > 0$ find a matrix $\hat{M} = \hat{M}_k$ of rank less than k which minimizes $\sigma_{\max}(M - \hat{M})$.

Since the set of matrices of given dimensions of rank less than k is not convex (unless k is larger than one of the dimensions), one can expect that the matrix rank reduction problem will be difficult to solve. However, in fact it has a relatively simple computationally efficient solution.

Let

$$M = \sum_{r=1}^{m} u_r \sigma_r v_r'$$

be a singular value decomposition of M, which means that the families $\{u_r\}_{r=1}^m$ and $\{v_r\}_{r=1}^m$ are orthonormal, and $\{\sigma_r\}_{r=1}^m$ is a monotonically decreasing sequence of positive numbers, i.e.

$$u_i' u_r = v_i' v_r = \begin{cases} 1, & i = r, \\ 0, & i \neq r, \end{cases} \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m > 0.$$

Here m is the rank of M, the numbers σ_r are called *singular numbers* of M, and the vectors v_r, u_r are, respectively, *right* and *left* singular vectors of M. The vector v_1 has unit length $|v_1| = 1$ (the Euclidean norm) and yields a maximal (over the set of all vectors of unit length) length $\sigma_1 = |Mv_1|$ when transformed with M. Vector u_1 is defined by $Mv_1 = \sigma_1 u_1$. The vector v_2 has unit length, is orthogonal to v_1, and yields a maximal (over all vectors of unit length and orthogonal to v_1) length $\sigma_2 = |Mv_2|$ when transformed with M. Vector u_2 is defined by $Mv_2 = \sigma_2 u_2$. In general, the vector v_r has unit length, is orthogonal to $v_1, \ldots, v_{r-1}$, and yields a maximal (over all vectors of unit length and orthogonal to $v_1, \ldots, v_{r-1}$) length $\sigma_r = |Mv_r|$ when transformed with M. Vector u_r is defined by $Mv_r = \sigma_r u_r$.

Another useful interpretation of singular vectors v_r, u_r and singular numbers σ_r is by the eigenvalue decompositions

$$(M'M)v_r = \sigma_r^2 v_r, \quad (MM')su_r = \sigma_r^2 u_r.$$

An approximation $\hat{M} = \hat{M}_k$ of rank less than k which minimizes $\sigma_{\max}(M - \hat{M})$ is given by

$$\hat{M}_k = \sum_{r=1}^{k-1} u_r \sigma_r v_r'$$

and yields an approximation error of $\sigma_{\max}(M - \hat{M}_k) = \sigma_k$. In other words, matching the original linear transformation M on the first $k-1$ inputs of highest amplification yields the best reduced rank approximation $\hat{M}$.

In most cases there are *many* optimal approximations $\hat{M}$. For example, when

$$M = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then both

$$\hat{M}_a = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{M}_b = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

are optimal rank one approximations of M.

The framework of optimal matrix rank reduction can be easily extended to the class of linear transformations M from one Hilbert space to another. (In the case of a real n-by-m matrix M, the Hilbert spaces are $\mathbf{R}^m$ and $\mathbf{R}^n$.) One potential complication is that the vectors v_r of maximal amplification do not always exist. In these notes, the following definition of singular numbers $\sigma_r = \sigma_r(M)$ will be used: if the first k right singular vectors $v_1, \ldots, v_k$ of M can be defined, but the supremum

$$\rho = \sup\{|Mv| : \ |v| = 1, \ v \text{ orthogonal to } v_1, \ldots, v_k\}$$

is not achievable, then $\sigma_r(M) = \rho$ for all $r > k$.

8.2.4 SVD of a Hankel operator

Any causal stable LTI system with a state-space model

$$dx/dt = Ax + Bf, \quad y = Cx + Df$$

defines a map

$$\text{past input} \ \mapsto \ x(0)$$

To produce $x(0) = x_0$, the energy of the past input must be *at least* $x_0'\psi_0$, where ψ_0 is an arbitrary solution of $W_c\psi_0 = x_0$,

$$W_c = \int_0^\infty e^{At} B B' e^{A't} dt > 0$$

is the *controllability Grammian* of the system. (If W_c is not positive definite, and equation $W_c\psi_0 = x_0$ is not solvable with respect to ψ_0, x_0 is an unreachable state.) The lowest energy input $f(t)$ producing x_0 is given by

$$f(-t) = B' e^{A't} \psi_0 \quad (t > 0).$$

Note that calculation of W_c is easy via the Lyapunov equation

$$AW_c + W_c A' = -BB'$$

The energy of the future output produced by the initial state $x(0) = x_0$, provided zero input for $t > 0$ equals $x_0' W_o x_0$, where

$$W_0 = \int_0^\infty e^{A't} C' C e^{At} dt$$

is the observability Gamian of the system. The output produced by x_0 is given by

$$y(t) = Ce^{At} x_0,$$

and calculation of W_o is easy via the Lyapunov equation

$$W_o A + A' W_o = -C'C$$

SVD of a Hankel operator H can be expressed in terms of its Gramians:
Let w_i be the normalized eigenvectors of $R = W_c^{1/2} W_o W_c^{1/2}$, i.e.

$$Rw_i = \lambda_i w_i, \quad \lambda_1 \geq \lambda_2 \geq \ldots, \ \lambda_m > 0, \ \lambda_{m+1} = 0$$

The SVD of H is given by

$$H = \sum_{k=1}^{m} u_k \sigma_k v_k',$$

where $\sigma_k = \lambda_k^{1/2}$,

$$u_k(t) = Ce^{At} W_c^{1/2} w_k \lambda_k^{-1/2} \quad (t > 0)$$
$$v_k(t) = \lambda_k^{-1} B' e^{-A't} W_o W_c^{1/2} w_k \quad (t < 0)$$

8.2.5 Technical details of the proof

Let $M : \mathbf{R}^n \mapsto L^2(0, \infty)$ be defined by

$$(Mx_0)(t) = Ce^{At} x_0$$

and let $N : L^2(-\infty, 0) \mapsto \mathbf{R}^n$ be

$$Nu(\cdot) = \int_{-\infty}^{0} e^{-At} B u(t) dt$$

By the definition of W_c, W_o,

$$W_o = M'M, \quad W_c = NN'$$

Hence M, N can be represented in the form

$$M = UW_o^{1/2}, \quad N = W_c^{1/2}V'$$

where linear transformations U, V preserve the 2-norm.

Since the Hankel operator under consideration has the form

$$H = MN = UW_o^{1/2}W_c^{1/2}V'$$

in order to find SVD of H, it is sufficient to find SVD of

$$F = W_o^{1/2}W_c^{1/2}$$

Since $F'F = R$, the SVD is given by

$$W_o^{1/2}W_c^{1/2} = \sum_{k=1}^{m} \bar{u}_k \lambda_k^{1/2} \bar{v}_k'$$

where

$$u_k = W_o^{1/2}W_c^{1/2}w_k\lambda_l^{-1/2},$$
$$v_k = W_c^{1/2}W_o p^{1/2}w_k\lambda_k^{-1} = w_k.$$