6.253: Convex Analysis and Optimization
Midterm

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Problem 1

State which of the following statements are true and which are false. You don't have to justify your answers:

1. If X_1, X_2 are convex sets that can be separated by a hyperplane, and X_1 is open, then X_1 and X_2 are disjoint. (8 points)
   TRUE

2. If $f : \mathbf{R}^n \mapsto \mathbf{R}$ is a convex function that is bounded in the sense that for some $\gamma > 0, |f(x)| \leq \gamma$ for all $x \in \mathbf{R}^n$, then the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathbf{R}^n, \end{array}$$

   has a solution. (8 points)
   TRUE

3. The support function of the set $\{(x_1, x_2) \mid |x_1| + |x_2| = 1\}$ is $\sigma(y) = \max\{y_1, y_2\}$. (8 points)
   FALSE

4. If $f : \mathbf{R}^n \mapsto (-\infty, \infty]$ is a convex function such that $\partial f(\bar{x})$ is nonempty for some $\bar{x} \in \mathbf{R}^n$, then f is lower semicontinuous at $\bar{x}$. (8 points)
   TRUE

5. If $M = \{(u, w) \mid u \in \mathbf{R}, |u| \leq w\}$, the dual function in the MC/MC framework corresponding to M is $q(\mu) = 0$ for all $\mu \in \mathbf{R}$. (8 points)
   FALSE

6. Let $f : \mathbf{R}^n \mapsto (-\infty, \infty]$ be convex and S be a subspace. If $\bar{x} \in S$ and $\partial f(\bar{x}) \cap S^\perp \neq \emptyset$, then $\bar{x}$ minimizes f over S. (8 points)
   TRUE

Problem 2

Consider the two-dimensional problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0 \\ & x \in X, \end{array}$$

where

$$f(x) = e^{-x_1}, \quad g(x) = \frac{x_1^2}{x_2}, \quad X = \{(x_1, x_2) \,|\, x_2 > 0\}.$$

1. Is the function g convex over X? (10 points)
2. Plot the set $\overline{M} = \{(u, w) \,|\, \exists\, x \in X \text{ such that } g(x) \leq u,\ f(x) \leq w\}$. (15 points)
3. What is the optimal value f^* of the problem? (5 points)
4. What is the optimal value of the dual and the duality gap? (15 points)
5. Consider the perturbed problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq u \\ & x \in X, \end{array}$$

   for $u > 0$. Is there a duality gap? (7 points)

Solution.

1. The function g is convex over X. Its Hessian is:
   $\nabla^2 g(x) = \begin{bmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \\ \frac{-2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix} = \frac{2}{x_2^3} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}^T \geq 0$. Therefore, according to Proposition 1.1.10, g is convex over X.
2. For $u < 0$, there is no $x \in X$such that $g(x) \leq u$.
   For $u = 0$, we have $x_1 = 0, f(x) = 1$. Therefore, $w \geq 1$.
   For $u > 0$, we have $\frac{x_1^2}{x_2} \leq u \Leftrightarrow x_1 \leq \pm\sqrt{ux_2}$. Therefore, x_1 changes from $(0, \infty)$, and $w > 0$. Therefore, $\overline{M}$ consists of the positive orphant and the halfline $\{(0, w) \,|\, w \geq 1\}$.
3. From the description of $\overline{M}$, we can see that $f^* = 1$.
4. From the description of $\overline{M}$, we can see that the dual function is $q(\mu) = \begin{cases} 0 & if \mu \geq 0 \\ -\infty & if \mu < 0 \end{cases}$
   Therefore, $q^* = 0$ and there is a duality gap of $f^* - q^* = 1$. Clearly, Slater's condition doesn't hold for this problem.
5. For the perturbed problem, Slater's condition holds, therefore we have no duality gap.