6.254 : Game Theory with Engineering Applications
Lecture 10: Evolution and Learning in Games

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March 9, 2010

Outline

- Myopic and Rule of Thumb Behavior
- Evolution
- Evolutionarily Stable Strategies
- Replicator Dynamics
- Learning in Games
- Fictitious Play
- Convergence of Fictitious Play

- Reading:
- Osborne, Chapter 13.
- Fudenberg and Levine, Chapters 1 and 2

Motivation

- Do people play Nash equilibrium?
- In class, in the context of the k-beauty game, we saw that even very smart MIT students do not play the unique Nash equilibrium (or the unique strategy profile surviving iterated elimination of strictly dominated strategies).
- Why?
  - Either because in new situations, it is often quite complex to work out what is "best".
  - Or more likely, because, again in new situations, individuals are uncertain about how others will play the game.
- If we played the k-beauty game several more times, behavior would have approached or in fact reached the Nash equilibrium prediction.

Motivation (continued)

- This reasoning suggests the following:
- Perhaps people behave using simple rules of thumb; these are somewhat "myopic," in the sense that they do not involve full computation of optimal strategies for others and for oneself.
- But they are also "flexible" rules of thumb in the sense that they adapt and respond to situations, including to the (actual) behavior of other players.
- What are the implications of this type of adaptive behavior?
- Two different and complementary approaches:
  1. Evolutionary game theory.
  2. Learning in games.

Evolution and Game Theory

- The theory of evolution goes back to Darwin's classic, *The Origins of Species* (and to Wallace).

- Darwin focused mostly on evolution and adaptation of an organism to the environment in which it was situated. But in *The Descent of Man*, in the context of sexual selection, he anticipated many of the ideas of evolutionary game theory.

- Evolutionary game theory was introduced by John Maynard Smith in *Evolution and the Theory of Games*, and in his seminal papers, Maynard Smith (1972) "Game Theory and the Evolution of Fighting" and Maynard Smith and Price (1973) "The Logic of Animal Conflict".

- The theory was formulated for understanding the behavior of animals in game-theoretic situations (to a game theorist, **all** situations). But it can equally well be applied to modeling "myopic behavior" for more complex organisms—such as humans.

Evolution in Strategies

- In its simplest form the story goes like this: each organism is born programmed to play a particular strategy.

- The game is the **game of life**—with payoffs given as fitness (i.e., expected number of offsprings). If the organism is successful, it has greater fitness and more offspring, also programmed to play in the same way. If it is unsuccessful, it likely dies without offspring.

- **Mutations** imply that some of these offspring will randomly play any one of the feasible strategies.

- This situation can then be modeled in two different ways:

  1. By defining a concept of equilibrium appropriate for this evolutionary "competition". The concept that Maynard Smith proposed is evolutionary stability.
  2. By defining the dynamics of evolution more explicitly through replicator dynamics.

The Setting

- Consider a large population of agents (organisms, animals, humans).

- At each instant, each agent is **randomly matched** with one other agent, and they play a symmetric strategic form game.

- Each agent is programmed (committed to) to playing a given strategy.

- The interaction between each pair of organisms can be modeled as a two player symmetric game $\langle S, u \rangle$ with a common set S of actions and a payoff function u.
  - For any pair of actions (s, s'), the payoff $u(s, s')$ is interpreted as the fitness of an organism that takes the action s when his opponent takes the action s'.

- Strategies with higher payoffs expand and those with lower payoffs contract.

- We assume that the adherents of a strategy multiply at a rate proportional to their payoff and look for a a set of strategies that is stable in the sense that it cannot be "invaded" by any other strategy:
  - if a small fraction of the population consists of identical mutants, every mutant gets an expected payoff lower than that of any nonmutant.

A Reminder and A New Concept

Definition

(Nash equilibrium) *A (pure strategy) Nash Equilibrium of a strategic form game* $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ *is a strategy profile* $s^* \in S$ *such that for all* $i \in \mathcal{I}$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \qquad \text{for all } s_i \in S_i.$$

Definition

(Strict Nash equilibrium) *A strict Nash Equilibrium of a strategic form game* $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ *is a strategy profile* $s^* \in S$ *such that for all* $i \in \mathcal{I}$

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*) \qquad \text{for all } s_i \in S_i.$$

- Clearly, a strict Nash equilibrium may not exist.
- Also, a strict Nash equilibrium is necessarily a pure strategy equilibrium (why?).

The Canonical Game

- The canonical game used in much of evolutionarily game theory to motivate the main ideas is the Hawk-Dove game:

Player 1/Player 2	Hawk	Dove
Hawk	$\left(\frac{1}{2}\left(v-c\right), \frac{1}{2}\left(v-c\right)\right)$	$(v, 0)$
Dove	$(0, v)$	$\left(\frac{1}{2}v, \frac{1}{2}v\right)$

- Interpretation:
  - There is a resource of value v to be shared. If a player plays "Hawk," it is aggressive and will try to take the whole resource for itself. If the other player is playing "Dove," it will succeed in doing so. If both players are playing "Hawk," then they fight and they share the resource but lose c in the process. If they are both playing "Dove," then they just share the resource.

- Interpret the payoffs as corresponding to **fitness**—e.g., greater consumption of resources leads to more offspring.

The Canonical Game (continued)

- Depending on the value of c relative to v, there are different types of equilibria.
  - If $v > c$, then there is a unique **strict** Nash equilibrium, which is (Hawk, Hawk).
  - If $v = c$, then there exists a unique Nash equilibrium, (Hawk, Hawk), though this is not a strict Nash equilibrium.
  - If $v < c$, then there exists three Nash equilibria: (Hawk, Dove) and (Dove, Hawk), which are non-symmetric strict equilibria, and a mixed strategy symmetric equilibrium.

Evolution in the Hawk-Dove Game

- If $v > c$, then we expect all agents to choose "Hawk". Those who do not will have lower fitness.

- A different way of thinking about the problem: imagine a population of agents playing "Dove" in this case.

- Suppose there is a **mutation**, so that one agent (or a small group of agents) starts playing "Hawk".

- This latter agent and its offspring will **invade** the population, because they will have greater fitness.

- The notion of evolutionarily stable strategies or evolutionary stability follows from this reasoning.

Evolutionarily Stable Strategies – Intuition

- In a symmetric two-player game, under what conditions is the strategy σ^* evolutionarily stable?

- Suppose that a small group of mutants choosing a strategy different from σ^* enters the population.

- Denote the fraction of mutants in the population by ϵ and assume that the mutant adopts the strategy σ.

- The expected payoff of a mutant is:

$$(1 - \epsilon)u(\sigma, \sigma^*) + \epsilon u(\sigma, \sigma).$$

- The expected payoff of an organism that adopts the strategy σ^* is

$$(1 - \epsilon)u(\sigma^*, \sigma^*) + \epsilon u(\sigma^*, \sigma).$$

- For any mutation to be driven out of the population, we need the expected payoff of any mutant to be less than the expected payoff of a normal organism:

$$(1 - \epsilon)u(\sigma^*, \sigma^*) + \epsilon u(\sigma^*, \sigma) > (1 - \epsilon)u(\sigma, \sigma^*) + \epsilon u(\sigma, \sigma).$$

Evolutionarily Stable Strategies

- Consider a two player, symmetric strategic form game, so we write it simply as $\langle S, u \rangle$. A (possibly mixed) strategy is $\sigma \in \Sigma$.

Definition

(Evolutionarily stable strategy I) *A strategy $\sigma^* \in \Sigma$ is evolutionarily stable if there exists $\bar{\varepsilon} > 0$ such that for any $\sigma \neq \sigma^*$ (naturally with $\sigma \in \Sigma$) and for any $\varepsilon < \bar{\varepsilon}$, we have*

$$u(\sigma^*, \varepsilon\sigma + (1 - \varepsilon)\sigma^*) > u(\sigma, \varepsilon\sigma + (1 - \varepsilon)\sigma^*). \qquad \text{(Condition I)}$$

- Interpretation: strategy σ^* is evolutionarily stable if it cannot be invaded by any $\sigma \neq \sigma^*$, i.e., if, starting with a population playing σ^*, a small fraction $\varepsilon < \bar{\varepsilon}$ of agents play σ, then these players do worse (have lower fitness) than those playing σ^*.

Evolutionary Stability: Alternative Definition

- An alternative definition is:

Definition

(Evolutionarily stable strategy II) *A strategy $\sigma^* \in \Sigma$ is evolutionarily stable if for any $\sigma \neq \sigma^*$ (with $\sigma \in \Sigma$), we have*

  ●

$$u(\sigma^*, \sigma^*) \geq u(\sigma, \sigma^*).$$

① *Moreover, if, for some $\sigma \in \Sigma$, $u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*)$, then*

$$u(\sigma^*, \sigma) > u(\sigma, \sigma).$$

- Interpretation: An evolutionarily stable strategy σ^* is a Nash equilibrium. If σ^* is not a strict Nash equilibrium, then any other strategy σ that is a best response to σ^* must be worse against itself than against σ^*.

Evolutionary Stability: Equivalence of the Two Definitions

Theorem

The two definitions of evolutionarily stable strategies are equivalent.

Proof: (First implies second)

- Since the first definition holds for any $\varepsilon < \bar{\varepsilon}$, as $\varepsilon \to 0$,
  $u(\sigma^*, \varepsilon\sigma + (1 - \varepsilon)\sigma^*) > u(\sigma, \varepsilon\sigma + (1 - \varepsilon)\sigma^*)$ implies

  $$u(\sigma^*, \sigma^*) \geq u(\sigma, \sigma^*),$$

  thus establishing part 1 of the second definition.

Proof (continued)

- To establish part 2, suppose that $u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*)$. Recall that u is linear in its arguments (since it is *expected* utility), so Condition I, $u(\sigma^*, \varepsilon\sigma + (1 - \varepsilon)\,\sigma^*) > u(\sigma, \varepsilon\sigma + (1 - \varepsilon)\,\sigma^*)$, can be written as

$$\varepsilon u(\sigma^*, \sigma) + (1 - \varepsilon)\,u(\sigma^*, \sigma^*) > \varepsilon u(\sigma, \sigma) + (1 - \varepsilon)\,u(\sigma, \sigma^*).$$

- Since $u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*)$, this is equivalent to

$$\varepsilon u(\sigma^*, \sigma) > \varepsilon u(\sigma, \sigma),$$

Since $\varepsilon > 0$, part 2 of the second definition follows.

Proof (continued)

(Second implies first)

- We have that for any $\sigma \in \Sigma$,

$$u(\sigma^*, \sigma^*) \geq u(\sigma, \sigma^*).$$

- If the inequality is strict, then Condition I in the first definition

$$\varepsilon u(\sigma^*, \sigma) + (1 - \varepsilon) u(\sigma^*, \sigma^*) > \varepsilon u(\sigma, \sigma) + (1 - \varepsilon) u(\sigma, \sigma^*), \qquad (*)$$

  is satisfied for $\epsilon = 0$, and hence for sufficiently small ϵ as well.

- If this relation holds as equality, then the second definition implies

$$u(\sigma^*, \sigma) > u(\sigma, \sigma).$$

  Multiply this by ε, use the fact that $u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*)$, and add
  $(1 - \varepsilon) u(\sigma^*, \sigma^*)$ to the left hand side and $(1 - \varepsilon) u(\sigma, \sigma^*)$ to the right
  hand side, which gives $(*)$ and hence Condition I.

Evolutionary Stability and Nash Equilibrium

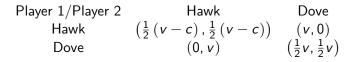- Now given our second definition, the following is immediate:

Theorem

1. *A strict (symmetric) Nash equilibrium of a symmetric game is an evolutionarily stable strategy.*

2. *An evolutionarily stable strategy is a Nash equilibrium.*

Proof: Both parts immediately follow from the second definition.

- Their converses are not true, however, as we will see.

Monomorphic and Polymorphic Evolutionarily Stability

- In addition, we could require an evolutionarily stable strategy (ESS) to be *monomorphic*—that is, all agents to use the same (pure) strategy.
- The alternative is *polymorphic*, where different strategies coexist, mimicking a mixed strategy equilibrium.
- With these definitions, let us return to the Hawk-Dove game.

The Hawk-Dove Game

Player 1/Player 2	Hawk	Dove
Hawk	$\left(\frac{1}{2}\left(v-c\right),\frac{1}{2}\left(v-c\right)\right)$	$(v,0)$
Dove	$(0,v)$	$\left(\frac{1}{2}v,\frac{1}{2}v\right)$

- Recall that if $v > c$, then there is a unique **strict** Nash equilibrium, which is (Hawk, Hawk). Therefore, in this case "Hawk" is also an evolutionarily stable strategy.
- Moreover, it is monomorphic.

The Hawk-Dove Game (continued)

- What happens if $v = c$?
- Recall that now there is a unique Nash equilibrium, (Hawk, Hawk), which is not a strict Nash equilibrium.
- We will now show that it is still an evolutionarily stable strategy.
- Since "Dove" is also a best response, we need to look at $u(H, D)$ vs. $u(D, D)$. Clearly the first one is greater, so part 2 of the second definition is satisfied. Therefore "Hawk" is evolutionarily stable.
    - It is also monomorphic.
- This example also shows that strict Nash equilibrium is stronger than evolutionarily stable strategy.

The Hawk-Dove Game (continued)

- Suppose now $v < c$, then there exists three Nash equilibria: (Hawk, Dove) and (Dove, Hawk), which are non-symmetric strict equilibria, and a mixed strategy symmetric equilibrium.

- The first observation is that there exists no monomorphic evolutionarily stable strategy. This shows the importance of looking at polymorphic strategies.

- Since this is a random matching game, clearly the non-symmetric equilibria are irrelevant (why?).

- Is the mixed strategy equilibrium evolutionarily stable?

The Hawk-Dove Game (continued)

- First note that when $v < c$, the unique mixed strategy equilibrium of the strategic form game involves each player playing "Hawk" with probability v/c. The polymorphic evolutionary stable outcome will be a population where fraction v/c of the agents are type "Hawk". Let us designate this by σ^*.

- We now need to show that such an outcome cannot be invaded by any other (mixed) strategy. That is, we need to check part 2 of the second definition. (Clearly, since σ^* is a mixed strategy equilibrium, any other mixed strategy σ satisfies $u(\sigma^*, \sigma^*) \geq u(\sigma, \sigma^*)$).

- Consider a mixed strategy, $\sigma \neq \sigma^*$, where a fraction $p \neq v/c$ play "Hawk".

The Hawk-Dove Game (continued)

- Then

$$
\begin{aligned}
u(\sigma, \sigma) &= p^2 \times \frac{1}{2}(v - c) + p(1 - p) \times v \\
&\quad + p(1 - p) \times 0 + (1 - p)^2 \times \frac{1}{2}v \\
u(\sigma^*, \sigma) &= \frac{v}{c}p \times \frac{1}{2}(v - c) + \frac{v}{c}(1 - p) \times v \\
&\quad + \left(1 - \frac{v}{c}\right)p \times 0 + \left(1 - \frac{v}{c}\right)(1 - p) \times \frac{1}{2}v
\end{aligned}
$$

- Therefore

$$
u(\sigma^*, \sigma) - u(\sigma, \sigma) = \frac{1}{2}c\left(\frac{v}{c} - p\right)^2 > 0,
$$

  which establishes the desired result.

- This result also shows the possibility of polymorphic evolutionarily stable strategies.

Nash Equilibrium Does Not Imply ESS

- Consider the modified rock-paper-scissors game:

$$
\begin{array}{cccc}
 & R & P & S \\
R & (\gamma, \gamma) & (-1, 1) & (1, -1) \\
P & (1, -1) & (\gamma, \gamma) & (-1, 1) \\
S & (-1, 1) & (1, -1) & (\gamma, \gamma)
\end{array}
$$

- Here $0 \leq \gamma < 1$. If $\gamma = 0$, this is the standard rock-paper-scissors game.

- For all such γ, there is a unique mixed strategy equilibrium $\sigma^* = (1/3, 1/3, 1/3)$, with expected payoff $u(\sigma^*, \sigma^*) = \gamma/3$. But for $\gamma > 0$, this is not ESS. For example, $\sigma = R$ would invade, since $u(\sigma, \sigma^*) = \gamma/3 < u(\sigma, \sigma) = \gamma$.

- This also shows that ESS doesn't necessarily exist.

Do Animals Play Games?

- The answer seems to be yes.
- They seem to play mixed strategies: sticklebacks are able to coordinate between the two sides of a fish tank, with different amounts of food "supply".
- This is like a "mixed strategy," since any food is shared among the sticklebacks at that end.
- Remarkably, when the relative amounts of food supplies into the fish tank at two sides is varied, sticklebacks are able to settle into the appropriate "mixed strategy" pattern given the new food supplies.

Do Animals Play Games?

- Side-blotched lizards seem to play a version of the Hawk-Dove game. Three productive strategies for male lizards with distinct throat colors (that are genetically determined):
  - orange color: very aggressive and defend large territories;
  - blue color: less aggressive defense smaller territories;
  - yellow color: not aggressive, opportunistic mating behavior.
- Tails seem to be as follows:
  - when all are orange, yellow does well; when all are yellow, blue does well; and when all are blue, orange does well.
- This is similar to the modified rock-paper-scissors pattern, and in nature, it seems that there are fluctuations in composition of male colorings as we should expect on the basis that the game does not have any evolutionarily stable strategies.
- See Karl Sigmund (1993) *Games of Life*, for many more fascinating examples, for animals, lower organisms and cells.

Dynamics

- The discussion of "dynamics" so far was largely heuristic.
- Are there actual dynamics of populations resulting from "game-theoretic" interactions that lead to evolutionarily stable strategies?
- Question at the intersection of game theory and population dynamics.
- The answer to this question is yes, and here we will discuss the simplest example, replicator dynamics.
- Throughout, we continue to focus on symmetric games.

Replicator Dynamics

- Let us formalize the discussion of fitness and offspring provided above.
- Let us enumerate the strategies by $s = 1, 2, ..., K$.
- Denote the fraction of the population playing strategy s by x_s.
- The setup is similar to that considered above: at each instant, each agent is randomly matched with another from a large population.
- What matters is expected fitness given by

$$u(s, \sigma).$$

In particular, recall that this is the expected fitness (payoff) of agents playing s when the mixed strategy induced by the polymorphic strategy profile is σ.

Replicator Dynamics (continued)

- Then, we can posit the following dynamic evolution:

$$x_s\left(t+\tau\right) - x_s\left(t\right) = x_s\left(t\right)\frac{\tau\left[u\left(s,\sigma\left(t\right)\right) - \bar{u}\left(\sigma\left(t\right)\right)\right]}{\bar{u}\left(\sigma\left(t\right)\right)},$$

(Replicator equation)

for each $s = 1, 2, ..., K$ and for all t and τ, where

$$\bar{u}\left(\sigma\left(t\right)\right) = \sum_{s=1}^{K} x_s\left(t\right) u\left(s,\sigma\left(t\right)\right)$$

is average fitness at time t and $\sigma\left(t\right)$ is the vector of $x_s\left(t\right)$'s.

- Naturally, $\sum_{s=1}^{K} x_s\left(t\right) = 1$ by definition.
- This equation gives discrete time dynamics when $\tau = 1$. But the equation is valid for any τ.
- Intuitively, the greater is the fitness of a strategy relative to the average fitness, the greater is its relative increase in the population. Clearly, this equation is meaningful, i.e., $\sum_{s=1}^{K} x_s\left(t+\tau\right) = 1$.

Replicator Dynamics: Continuous Time

- It is most convenient to work with replicator dynamics in continuous time.
- Divide both sides of the replicator equation by τ and take the limit as $\tau \to 0$. This gives

$$\lim_{\tau \to 0} \frac{x_s(t+\tau) - x_s(t)}{\tau} = x_s(t) \frac{[u(s, \sigma(t)) - \bar{u}(\sigma(t))]}{\bar{u}(\sigma(t))}.$$

Therefore

$$\dot{x}_s(t) = x_s(t) \frac{u(s, \sigma(t)) - \bar{u}(\sigma(t))}{\bar{u}(\sigma(t))}, \qquad \text{(Continuous replicator)}$$

where recall that $\dot{x}_s(t) \equiv dx_s(t)/dt$.

- Notice that $x_s(t)$ is not written in the denominator of the left-hand side, since it can be equal to zero.

Replicator Dynamics: Implications

- Now we can think of the dynamics starting from an arbitrary distribution of strategies in the population.
- There are two ways of doing this:

  1. Ask whether a particular vector of distribution x^* is a **stationary state** of equation (Continuous replicator), meaning that it has $\dot{x}_s^*(t) = 0$ for all s.

  2. Ask whether a particular vector of distribution x^* is an **asymptotically stable state**, meaning that there exists a neighborhood of x^* such that starting from any x_0 in this neighborhood, dynamics induced by (Continuous replicator) approach x^*.

Replicator Dynamics and Nash Equilibria

Theorem

If x^* is a Nash equilibrium, then it is a stationary state.

Proof: If x^* is a Nash equilibrium, then it is a best response to itself, and thus no strategy has $u\left(s, \sigma\left(t\right)\right) - \bar{u}\left(\sigma\left(t\right)\right) > 0$, and $u\left(s, \sigma\left(t\right)\right) - \bar{u}\left(\sigma\left(t\right)\right) = 0$ only for strategies in the support of the mixed strategy profile induced by x^*. Thus for any s, either $u\left(s, \sigma\left(t\right)\right) - \bar{u}\left(\sigma\left(t\right)\right) = 0$ or $x_s\left(t\right) = 0$, and hence $\dot{x}_s^*\left(t\right) = 0$ for all s.

- However, the converse of this statement is not true, since if x^* corresponds to a non-Nash pure strategy, then $x_s^*\left(t\right) = 0$ for all s other than the pure strategy in question, and x^* is stable.
- Thus stability is not a particularly relevant concept. We would like x^* to be robust to "perturbations"—or against attempts at invasion. This requires *asymptotic stability*.

Replicator Dynamics and Nash Equilibria (continued)

Theorem

If x^* is asymptotically stable, then it is a Nash equilibrium.

- The proof is immediate if x^* corresponds to a pure strategy (monomorphic population).
- In the case where x^* corresponds to a mixed strategy Nash equilibrium, the proof is also straightforward but long. The basic idea is that equation (Continuous replicator) implies that we are moving in the direction of "better replies"—relative to the average. If this process converges, then there must not exist any more (any other) strict better replies, and thus we must be at a Nash equilibrium.
- The converse is again not true.

Replicator Dynamics and Nash Equilibria (continued)

- Consider, for example,

$$
\begin{array}{ccc}
 & A & B \\
A & (1,1) & (0,0) \\
B & (0,0) & (0,0)
\end{array}
$$

- Here (B,B) is a Nash equilibrium, but clearly it is not asymptotically stable, since B is weakly dominated, and thus any perturbation away from (B,B) will start a process in which the fraction of agents playing A steadily increases.

Replicator Dynamics and Evolutionary Stability

- The key result here is the following, which justifies the focus on Evolutionary stable strategies.

Theorem

If x^* is evolutionarily stable, then it is asymptotically stable.

- The proof is again somewhat delicate, but intuitively straightforward. The first definition of ESS states that for small enough perturbations, the evolutionarily stable strategy is a strict best response. This essentially implies that in the neighborhood of the ESS σ^*, σ^* will do better than any other strategy σ, and thus according to (Continuous replicator), the fraction of those playing σ^* should increase, thus implying asymptotic stability.

Evolution and Network Structure

- So far, no network structure in evolutionary interactions because of the **random matching** assumption (and this will be also the case when we turn to learning next).

- But this is not realistic. In practice, animals, organisms and humans play and compete more against "nearby" agents.

- One interesting area is to incorporate network structure into dynamics of game-theoretic behavior.

Learning vs. Evolution

- Evolution is a good model for fully myopic behavior. But even when individuals follow rules of thumb, they are not fully myopic.

- Moreover, in evolution, the time scales are long. We need "mutations," which are random and, almost by definition, rare.

- In most (human) game-theoretic situations, even if individuals are not fully rational, they can imitate more successful strategies quickly, and learn the behavior of their opponents and best respond to those.

- This suggests a related but distinct approach to dynamic game-theoretic behavior, which is taken in the literature on **learning in games**.

    - Note that this is different from Bayesian game-theoretic learning, which we will discuss later in the course.

Models of Learning

- One approach is to try to import menu of the insides of evolutionary game theory into the area of "learning in games".
- For example, for a symmetric game, we could posit **an imitation rule** that takes the form of equation (Continuous replicator), i.e., individuals imitate the strategies of others in proportion to how much they outperform the average in the population. Though plausible, this requires "global knowledge" on the part of individuals about others' payoffs.
- More importantly, in the context of learning, it may be more fruitful to ask: "what are players learning about?" The most plausible answer is the strategies of others.
- One of the earliest learning rules, fictitious play, introduced in Brown (1951) "Iterative solutions of games by fictitious play," is motivated by this type of reasoning.
- The idea is to look at a dynamic process where each player best responds to the time average of the behavior of its opponents.

Setup

- Let us first focus on a strategic form game $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$.
- The players play this game at times $t = 1, 2, \ldots$.
- The stage payoff of player i is again given by $u_i(s_i, s_{-i})$ (for pure strategy profile s_i, s_{-i}).
- For $t = 1, 2, \ldots$ and $i = 1, 2$, define the function $\eta_i^t : S_{-i} \to \mathbb{N}$, where $\eta_i^t(s_{-i})$ is the number of times player i has observed the action s_{-i} before time t. Let $\eta_i^0(s_{-i})$ represent a starting point (or fictitious past).
- For example, consider a two player game, with $S_2 = \{U, D\}$. If $\eta_1^0(U) = 3$ and $\eta_1^0(D) = 5$, and player 2 plays U, U, D in the first three periods, then $\eta_1^3(U) = 5$ and $\eta_1^3(D) = 6$.

The Basic Idea

- The basic idea of fictitious play is that each player assumes that his opponent is using a *stationary mixed strategy*, and updates his beliefs about this stationary mixed strategies at each step.

- Players choose actions in each period (or stage) to maximize that period's expected payoff given their prediction of the distribution of opponent's actions, which they form according to:

$$\mu_i^t(s_{-i}) = \frac{\eta_i^t(s_{-i})}{\sum_{\bar{s}_{-i} \in S_{-i}} \eta_i^t(\bar{s}_{-i})}.$$

- For example, in a two player game, player i forecasts player $-i$'s strategy at time t to be the empirical frequency distribution of past play.

Fictitious Play Model of Learning

- Given player i's belief/forecast about his opponents play, he chooses his action at time t to maximize his payoff, i.e.,

$$s_i^t \in \arg \max_{s_i \in S_i} u_i(s_i, \mu_i^t).$$

- Even though fictitious play is "belief based," it is also **myopic**, because players are trying to maximize current payoff without considering their future payoffs. Perhaps more importantly, they are also not learning the "true model" generating the empirical frequencies (that is, how their opponent is actually playing the game).

Example

- Consider the fictitious play of the following game:

$$
\begin{array}{ccc}
 & L & R \\
U & (3,3) & (0,0) \\
D & (4,0) & (1,1)
\end{array}
$$

- Note that this game is dominant solvable (D is a strictly dominant strategy for the row player), and the unique NE (D, R).
- Assume that $\eta_1^0 = (3, 0)$ and $\eta_2^0 = (1, 2.5)$. Then fictitious play proceeds as follows:
  - *Period 1:* Then, $\mu_1^0 = (1, 0)$ and $\mu_2^0 = (1/3.5, 2.5/3.5)$, so play follows $s_1^0 = D$ and $s_2^0 = L$.
  - *Period 2:* We have $\eta_1^1 = (4, 0)$ and $\eta_2^1 = (1, 3.5)$, so play follows $s_1^1 = D$ and $s_2^1 = R$.
  - *Period 3:* We have $\eta_1^1 = (4, 1)$ and $\eta_2^1 = (1, 4.5)$, so play follows $s_1^2 = D$ and $s_2^2 = R$.
  - *Periods 4:...*

Example (continued)

- Since D is a dominant strategy for the row player, he always plays D, and μ_2^t converges to $(0, 1)$ with probability 1.
- Therefore, player 2 will end up playing R.
- The remarkable feature of the fictitious play is that players don't have to know anything about their opponent's payoff. They only form beliefs about how their opponents will play.

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Spring 2010