Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.432 Stochastic Processes, Detection and Estimation

Problem Set 2

Spring 2004

Issued: Tuesday, February 10, 2004 **Due:** Thursday, February 19, 2004

Reading: For this problem set: Chapter 2, through Section 2.5.1
Next: Chapter 2, Chapter 3 through Section 3.2.4

Problem 2.1

Let x and y be statistically independent random variables with probability density functions

$$p_x(x) = \frac{1}{2}\delta(x+1) + \frac{1}{2}\delta(x-1)$$

and

$$p_y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right),$$

and let $z = x + y$, and $w = xy$.

(a) Find $p_z(z)$, the probability density function for z.

(b) Find the conditional probability density functions $p_{z|x}(z \mid x = -1)$ and $p_{z|x}(z|x = 1)$.

(c) Find the mean values $\bar{x} = m_x$ and $\bar{y} = m_y$, the variances σ_y^2 and σ_w^2, and the covariance λ_{yw}. Are y and w uncorrelated random variables? Are y and w statistically independent random variables?

(d) Are y and w Gaussian random variables? Are they jointly Gaussian? Explain.

Problem 2.2

Let x_1 and x_2 be zero-mean jointly Gaussian random variables with covariance matrix

$$\mathbf{\Lambda_x} = \begin{bmatrix} 34 & 12 \\ 12 & 41 \end{bmatrix}.$$

(a) Verify that $\mathbf{\Lambda_x}$ is a valid covariance matrix.

(b) Find the marginal probability density for x_1. Find the probability density for $y = 2x_1 + x_2$.

(c) Find a linear transformation defining two new variables

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \mathbf{P} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so that x_1' and x_2' are statistically independent and so that

$$\mathbf{P}\mathbf{P}^T = \mathbf{I},$$

where $\mathbf{I}$ is the 2×2 identity matrix.

Problem 2.3 (practice)

Let x be an N-dimensional zero-mean random vector whose covariance matrix has eigenvalues

$$\lambda_1 > \lambda_2 > \cdots > \lambda_N,$$

and corresponding eigenvectors

$$\phi_1, \phi_2, \cdots, \phi_N.$$

Suppose we wish to approximate x as a scalar random variable b times a deterministic N-dimensional vector $\mathbf{a}$ (i.e., $b\mathbf{a}$). This problem concerns finding the "best" b and $\mathbf{a}$.

(a) Let

$$b_{\text{opt}} = \arg\min_{b \in \mathbb{R}} \|\mathsf{x} - b\mathbf{a}\|,$$

where $\|\cdot\|$ is the Euclidean norm for $\mathbb{R}^N$, i.e., $\|\mathsf{z}\|^2 = \mathsf{z}^T\mathsf{z}$. Find an explicit expression for b_{opt} in terms of x and $\mathbf{a}$. Note that b_{opt} is a random variable.

(b) Let b_{opt} be as defined in part (a). Now define

$$\mathbf{a}_{\text{opt}} = \arg\min_{\mathbf{a} \in \mathbb{R}^N} E[\|\mathsf{x} - b_{\text{opt}}\mathbf{a}\|^2].$$

Show that

$$\mathbf{a}_{\text{opt}} = \arg\max_{\mathbf{a}} \frac{\text{var}\,[\mathbf{a}^T\mathsf{x}]}{\mathbf{a}^T\mathbf{a}}.$$

(c) Determine

$$\max_{\mathbf{a}} \frac{\text{var}\,[\mathbf{a}^T\mathsf{x}]}{\mathbf{a}^T\mathbf{a}}$$

and indicate the value(s) of $\mathbf{a}$ for which the maximum is achieved.

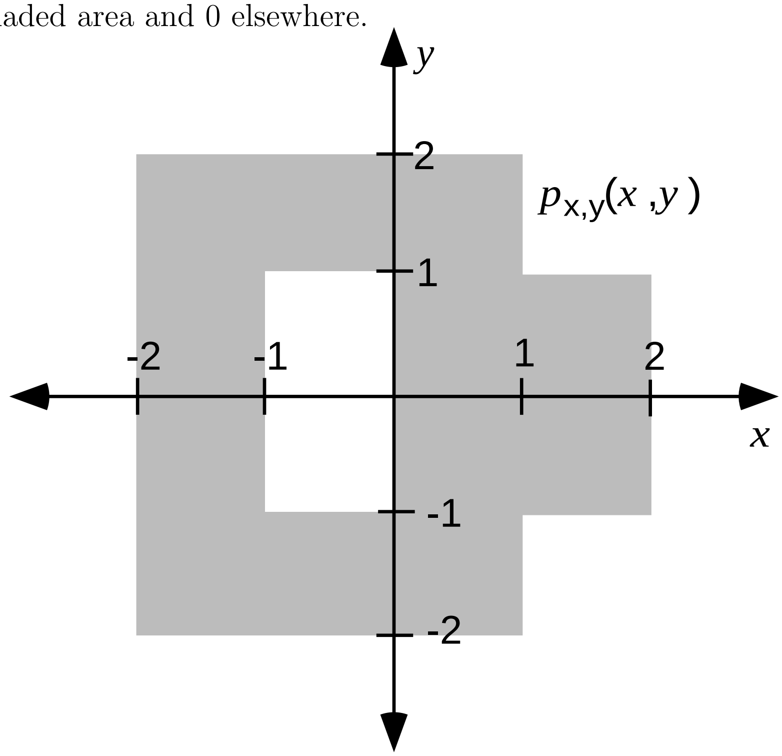
(d) Repeat part (c) when we impose the constraint that

$$\mathbf{a} \perp \phi_i \text{ (i.e., } \mathbf{a}^T \phi_i = 0\text{) for } i = 1, 2, \cdots, k-1,$$

for some $k \geq 2$. Again, indicate the value(s) of $\mathbf{a}$ for which the maximum is achieved.

Problem 2.4

Suppose x and y are random variables. Their joint density, depicted below, is constant in the shaded area and 0 elsewhere.



(a) Let $H = H_0$ when $\mathsf{x} \leq 0$, and let $H = H_1$ when $\mathsf{x} > 0$. Determine $P_0 = \Pr[H = H_0]$ and $P_1 = \Pr[H = H_1]$, and make fully labelled sketches of $p_{\mathsf{y}|H}(y|H_0)$ and $p_{\mathsf{y}|H}(y|H_1)$.

(b) Construct a rule $\hat{H}(y)$ for deciding between H_0 and H_1 given an observation $\mathsf{y} = y$ that minimizes the probability of error. Specify for which values of y your rule chooses H_1, and for which values it chooses H_0. That is, determine the regions

$$\begin{aligned} \mathcal{Z}_0 &= \{y \mid \hat{H}(y) = H_0\} \\ \mathcal{Z}_1 &= \{y \mid \hat{H}(y) = H_1\}. \end{aligned}$$

What is the resulting probability of error?

Problem 2.5
In the binary communication system shown in Fig. 5-1, messages $m = 0$ and $m = 1$ occur with a priori probabilities 1/4 and 3/4 respectively. Suppose that we observe r,

$$r = n + m,$$

where n is a continuous valued random variable with the pdf shown in Fig. 5-2. The random variable n is statistically independent of whether message $m = 0$ or $m = 1$ occurs.

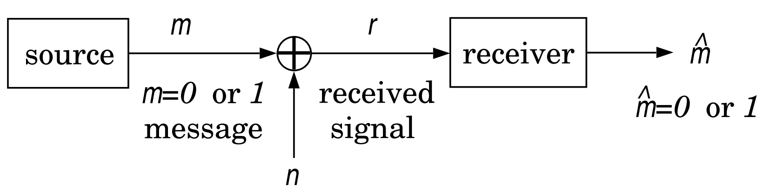


Figure 5-1

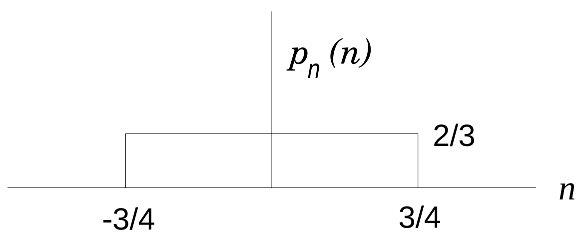


Figure 5-2

(a) Find the minimum probability of error detector, and compute the associated probability of error, $\Pr[\hat{m} \neq m]$.

(b) **(practice)** Suppose that the receiver does not know the a priori probabilities, so it decides to use a maximum likelihood (ML) detector. Find the ML detector and the associated probability of error. Is the ML detector unique? Justify your answer. If your answer is no, find a different ML receiver and the associated probability of error.

Problem 2.6
Consider the binary discrete-time communication system shown in Fig. 6-1. Assume that $m = 0$ and $m = 1$ are equally likely to occur. Under hypothesis H_m $(m = 0, 1)$, the received signal $r[n]$ is given by

$$r[n] = y_m[n] + w[n] = s_m[n] * h[n] + w[n], \qquad \text{for all } n,$$

where "$*$" denotes convolution, and where the $w[n]$'s are zero-mean statistically independent Gaussian random variables with variance σ^2. The two signals, $s_0[n]$ and

$s_1[n]$ are shown in Figs. 6-2 and 6-3, respectively. The parameter λ in Fig. 6-3 satisfies $0 \leq \lambda \leq 1$. The impulse response of the linear time invariant filter $h[n]$ is shown in Fig. 6-4.

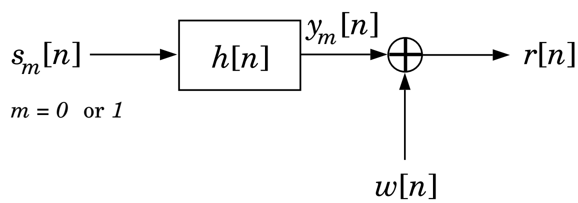


Figure 6-1

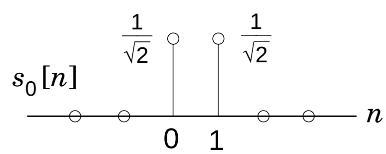


Figure 6-2

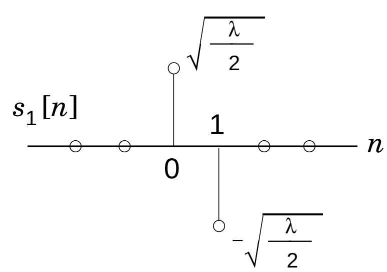


Figure 6-3

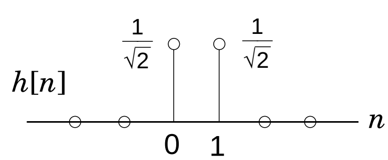


Figure 6-4

We wish to obtain a rule for making a decision about which hypothesis was used, based on the observed sequence $r[n]$.

(a) Which samples of $r[n]$ provide information in making the decision? Justify your reasoning.

(b) Find the minimum probability of error decision rule, based on observation of $r[n]$. Simplify your processor as much as possible to minimize computation.

(c) Obtain an expression for $\Pr[\varepsilon]$ in terms of λ and $\mathcal{Q}(\cdot)$, where $\mathcal{Q}(\cdot)$ is defined as

$$\mathcal{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\alpha^2/2} d\alpha.$$

(d) Find the values of λ (where $0 \leq \lambda \leq 1$) that minimize $\Pr[\varepsilon]$ for the detector obtained in part (b).

Problem 2.7 (practice)

Binary frequency shift keying (FSK) on a Rayleigh fading channel can be modeled in terms of a 4-dimensional observation vector

$$\mathsf{y} = \begin{bmatrix} \mathsf{y}_1 \\ \mathsf{y}_2 \\ \mathsf{y}_3 \\ \mathsf{y}_4 \end{bmatrix},$$

which is given by $\mathsf{y} = \mathsf{x} + \mathsf{z}$, where $\mathsf{z} \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$ and z is independent of x. Under $\mathsf{H} = H_0$,

$$\mathsf{x} = \begin{bmatrix} \mathsf{x}_1 \\ \mathsf{x}_2 \\ 0 \\ 0 \end{bmatrix},$$

whereas under $\mathsf{H} = H_1$,

$$\mathsf{x} = \begin{bmatrix} 0 \\ 0 \\ \mathsf{x}_3 \\ \mathsf{x}_4 \end{bmatrix}.$$

The x_i are independent, identically-distributed (IID) $N(0, \alpha^2)$ random variables. Furthermore, the two hypotheses are equally likely.

(a) Show that the ML receiver calculates $\mathsf{v}_0 = \mathsf{y}_1^2 + \mathsf{y}_2^2$ and $\mathsf{v}_1 = \mathsf{y}_3^2 + \mathsf{y}_4^2$, and chooses $\hat{\mathsf{H}} = H_0$ if $\mathsf{v}_0 \geq \mathsf{v}_1$, and chooses $\hat{\mathsf{H}} = H_1$ otherwise.

(b) Find $p_{\mathsf{v}_0|\mathsf{H}}(v_0|H_0)$ and $p_{\mathsf{v}_1|\mathsf{H}}(v_1|H_0)$.

(c) Let $\mathsf{u} = \mathsf{v}_0 - \mathsf{v}_1$, and find $p_{\mathsf{u}|\mathsf{H}}(u|H_0)$.

(d) Show that

$$\Pr[\varepsilon | \mathsf{H} = H_0] = \frac{1}{2 + \dfrac{\alpha^2}{\sigma^2}},$$

where ε denotes the error event $\hat{\mathsf{H}} \neq \mathsf{H}$. Explain why this is also the unconditional probability of an incorrect decision.

Problem 2.8
In the binary communications system shown below, messages $m = 0$ and $m = 1$ occur with a priori probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively. The random variable n is independent from m and takes on the values -1, 0, 1 with probabilities $\frac{1}{8}$, $\frac{3}{4}$, $\frac{1}{8}$ respectively

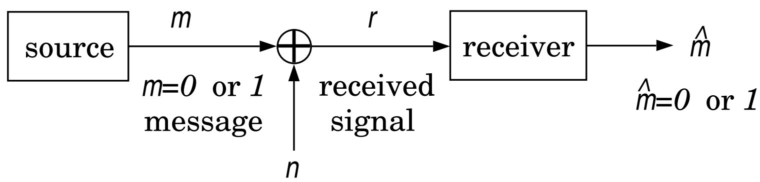


Figure 8-1

Find the receiver which achieves the maximum probability of correct decision. Compute the probability of error for this receiver.