MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J — Fall 2008

Problem Set 1 — due 9/8/2008

Readings:
(a) Notes from Lecture 1
(b) Handout on background material on sets and real analysis (Recitation 1).

Supplementary readings:
[GS], Sections 1.1-1.3.
[W], Sections 1.0-1.5, 1.9.

Exercise 1.

(a) Show that the union of countably many countable sets is countable.

(b) A real number x is rational if $x = m/n$, where m is an integer and n is a nonzero integer. Show that the set of rational numbers $\mathbb{Q}$ is countable.

Exercise 2. Let $\{x_n\}$ and $\{y_n\}$ be real sequences that converge to x and y, respectively. Provide a formal proof of the fact that $x_n y_n$ converges to xy.

Exercise 3. We are given a function $f : A \times B \to \Re$, where A and B are nonempty sets.

(a) Assuming that the sets A and B are finite, show that

$$\max_{x \in A} \min_{y \in B} f(x, y) \leq \min_{y \in B} \max_{x \in A} f(x, y).$$

(b) For general nonempty sets (not necessarily finite), show that

$$\sup_{x \in A} \inf_{y \in B} f(x, y) \leq \inf_{y \in B} \sup_{x \in A} f(x, y).$$

Exercise 4. Let $\{A_n\}$ be a sequence of sets. Show that $\lim_{n\to\infty} A_n = A$ if and only $\lim_{n\to\infty} I_{A_n}(\omega) = I_A(\omega)$ for all ω.

Exercise 5. (The union bound) Let $(\Omega, \mathcal{F})$ be a measurable space, and consider a sequence $\{A_i\}$ of $\mathcal{F}$-measurable sets, not necessarily disjoint. Show that

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Hint: Express $\cup_{i=1}^{\infty} A_i$ as a countable union of disjoint sets.

Exercise 6. Let $\Omega = \mathbb{N}$ (the positive integers), and let $\mathcal{F}_0$ be the collection of subsets of Ω that either have finite cardinality or their complement has finite cardinality. For any $A \in \mathcal{F}_0$, let $\mathbb{P}(A) = 0$ if A is finite, and $\mathbb{P}(A) = 1$ if A^c is finite.

(a) Show that $\mathcal{F}_0$ is a field but not a σ-field.

(b) Show that $\mathbb{P}$ is finitely additive on $\mathcal{F}_0$; that is, if $A, B \in \mathcal{F}_0$, and A, B are disjoint, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

(c) Show that $\mathbb{P}$ is not countably additive on $\mathcal{F}_0$; that is, construct a sequence of disjoint sets $A_i \in \mathcal{F}_0$ such that $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_0$ and $\mathbb{P}(\cup_{i=1}^{\infty} A_i) \neq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

(d) Construct a decreasing sequence of sets $A_i \in \mathcal{F}_0$ such that $\cap_{i=1}^{\infty} A_i = \emptyset$ for which $\lim_{n\to\infty} \mathbb{P}(A_i) \neq 0$.

6.436J / 15.085J Fundamentals of Probability
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