MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Fall 2008 15.085/6.436

Review Session

1 σ-fields

A σ-field is a collection $\mathcal{F}$ of subsets of the sample space Ω satisfying the following properties:

1. If $A_1, A_2, \ldots,$ all belong to $\mathcal{F}$, then so does $\bigcup_{i=1}^{\infty} A_i$.
2. If $A \in \mathcal{F}$, then $A^c \in F$.
3. $\emptyset \in \mathcal{F}$.

We have previously described a number of σ-fields:

1. The set of all subsets of the sample space Ω is always a σ-field.
2. The Borel σ-field of the real line. This is the smallest σ field containing the intervals.
3. Coin toss σ-field. This is the smallest σ-field containing all events decidable in finite time.

For precise definitions, see the lecture notes. Meanwhile, here are some other examples to keep in mind while solving problems:

1. Let $A \in \Omega$. The collection of sets $\{\emptyset, A, A^c, \Omega\}$ form a σ-field, regardless of A.
2. Let $A \in \Omega$. Let $\mathcal{F}$ be defined as the collection of sets B such that B or B^c contains A. Lets verify that this is indeed a σ field.

   The third property is satisfied: $\emptyset \in \mathcal{F}$ since $A \subset \Omega = \emptyset^c$. The second property is clearly satisfied.

   As for the first property, let us suppose we have some sequence $B_1, B_2, \ldots,$ in $\mathcal{F}$. Then, there are two possibilities. First, suppose $A \subset B_i$ for some index i. Then, $A \subset \bigcup_{i=1}^{\infty} B_i$, and $\bigcup_{i=1}^{\infty} B_i \in \mathcal{F}$. Second, suppose that for all i, A is not a subset of B_i. Since, each $B_i \in \mathcal{F}$, this means $A \subset B_i^c$ for all i. Then,

   $$A \subset \bigcap_{i=1}^{\infty} B_i^c.$$

But since

$$(\bigcup_{i=1}^{\infty} B_i)^c = \bigcap_{i=1}^{\infty} B_i^c,$$

we get that

$$A \subset (\bigcup_{i=1}^{\infty} B_i)^c,$$

and therefore $\bigcup_{i=1}^{\infty} B_i \in \mathcal{F}$.

3. All subsets B of the real line such that B or B^c is countable[1].

   The proof of this parallels the last example. If $B_1, B_2, \dots$ are all in $\mathcal{F}$, then either all are countable, or at least one is uncountable. If all are countable, then their union is countable. If at least one is uncountable, then its complement must be countable. Thus the intersections of B_i^c are countable, which shows that the union of B_i is in $\mathcal{F}$.

Two common mistakes relating to σ-fields are:

1. If $A \in \mathcal{F}$ and $B \subset A$, then $B \in \mathcal{F}$.

   This is incorrect. Take, for example, the example from above: $\mathcal{F}$ is the collection of sets B such that B or B^c contains A, with $A = \{1, 2\}$. Then, $\{1, 2, 3\} \in \mathcal{F}$, but $\{1\} \notin \mathcal{F}$.

   Generally, this would imply every set is in $\mathcal{F}$, since $\Omega \in \mathcal{F}$ necessarily, and every set is a subset of Ω.

2. The union of an uncountable collection of sets in $\mathcal{F}$ is in $\mathcal{F}$.

   This is not correct. Take any subset of the real line X. Then,

$$X = \bigcup_{x \in X} \{x\}.$$

   Now we know that each singleton $\{x\}$ is Borel measurable, but this does not imply that every subset X of the real line is Borel measurable.

Lets work out some simple exercises.

1. Suppose $B \in \mathcal{F}$. Let $\mathcal{F}'$ be defined as

$$\mathcal{F}' = \{A \bigcap B \mid A \in \mathcal{F}\}.$$

[1]Here "countable" includes finite sets.

Show that $\mathcal{F}'$ is a σ-field of subsets of B.

Solution: clearly, $\emptyset \in \mathcal{F}'$.

If $C \in \mathcal{F}'$, then $C = A \bigcap B$ for some $A \in \mathcal{F}$. Then, $A^c \in \mathcal{F}$, so that $C' = A^c \bigcap B \in \mathcal{F}'$. However, $C \bigcup C' = B$ and $C \bigcap C' = \emptyset$, so that $C' = C^c$. This shows that complements belong to $\mathcal{F}'$.

Finally, if $C_1, C_2, \ldots, \in \mathcal{F}'$, there must exist sets A_i such that

$$C_i = A_i \bigcap B.$$

Then, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, so that $(\bigcup_{i=1}^{\infty} A_i) \bigcap B \in F'$. However,

$$(\bigcup_{i=1}^{\infty} A_i) \bigcap B = \bigcup_{i=1}^{\infty} (A_i \bigcap B) = \bigcup_{i=1}^{\infty} C_i,$$

showing that $\mathcal{F}$ is closed under countable unions.

2. Suppose instead that requiring that $\mathcal{F}$ is closed under countable unions, we require that $\mathcal{F}$ is closed under countable *disjoint* unions. Will the result be the same?

   The answer is no. Let $\Omega = \{1, 2, 3, 4\}$ and let $\mathcal{F}$ be the empty-set, the whole space, and the collection of all two-element subets of Ω:

$$\mathcal{F} = \{\emptyset, \Omega, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

   Then, $\mathcal{F}$ is closed under disjoint unions, but not unions.

2 Probability

1. Can a discrete probability space contain an infinite number of *independent* events $A_1, A_2, \ldots$, each with probability $1/2$?

   The answer is no. Each point of the probability space would lie in one of the four sets $A_1 \bigcap A_2, A_1 \bigcap A_2^c, A_1^c \bigcap A_2, A_1^c \bigcap A_2^c$, and hence have probability at most $(1/2)^2$. We can do this for arbitrarily n, to get that each point has probability 0.

2. Suppose the sequence A_n of measurable sets is nearly disjoint in the sense that

$$P(A_n \bigcap A_m) = 0,$$

for $n \neq m$. If $A = \bigcup_{i=1}^{\infty} A_n$, show that

$$P(A) = \sum_{i=1}^{\infty} P(A_i).$$

Solution: define

$$B_i = A_i \bigcap A_1^c \bigcap A_2^c \bigcap \cdots \bigcap A_{i-1}^c.$$

Then:

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

and $P(B_i) = P(A_i)$, and B_i are disjoint.

3. Suppose $\Omega = \{1, 2, 3, 4, 5\}$, and $\mathcal{F}$ is the set of all subsets of Ω. Let $P(A) = |A|/5$ for all A. Show that if A, B are independent, then one of them must be the empty set or Ω.

   Solution: We have

$$\frac{P(A, B)}{5} = \frac{P(A)}{5} \frac{P(B)}{5},$$

   or

$$5|A \bigcap B| = |A||B|$$

   So $|A||B|$ must be divisible by 5. Since 5 is a prime, it follows that one of the cardinalities is 5 or 0.

6.436J / 15.085J Fundamentals of Probability
Fall 2008