MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Problem Set 10

Readings:
Notes from Lecture 18, 19 and 20.
[Cinlar] Chapter III.
[GS] Section 7.5, 7.10, 8.1-8.3.

Exercise 1. Let $S_n = \sum_{j=1}^{n} X_j$ be a sum of independent random variables X_j with $|X_j| \leq 1$ almost surely. Show that S_n converges in probability if and only if it converges almost surely (to a finite value).
(Hint: See how the case $\sum \mathrm{var}[X_j] = \infty$ was treated in the converse part of Kolmogorov-Khintchine in Lecture 19.)

Solution: Note that a.s. $\Rightarrow$ i.p. is instantaneous. So we only need i.p. $\Rightarrow$ a.s.

Let $S_n \to S$ in probability to a finite probability distribution; first, let's establish that we can WLOG assume that $E[X_j] = 0$ for all j. If $\sum_{j=1}^{\infty} E[X_j]$ converges, we can simply replace X_j with $\hat{X}_j := X_j - E[X_j]$ (the original S_n's converge if and only if the new $\hat{S}_n$'s converge). But $\sum_{j=1}^{\infty} E[X_j]$ cannot diverge because $|X_j| \leq 1$ and the X_j's are independent.

Now we show a.s. convergence via the Cauchy criterion

$$\lim_{n\to\infty} P[\sup_{k\geq 1} |S_{n+k} - S_n| > \varepsilon]$$

Since the X_i's are independent and mean-0, we can apply Kolmogorov's Inequality. In particular,

$$S_{n+k} - S_n = \sum_{j=n+1}^{n+k} X_j$$

so we get from Kolmogorov's Inequality that

$$P\Big[\sup_{k\geq 1} \Big|\sum_{j=n+1}^{n+k} X_j\Big| \geq \epsilon\Big] \leq \frac{2}{\varepsilon^2} \sum_{j=n+1}^{\infty} \mathrm{var}[X_j]$$

We'd like this to converge to 0 as $n \to \infty$ (then by the Cauchy criterion we're done); that happens if $\sum_{j=1}^{\infty} \mathrm{var}[X_j] = \sum_{j=1}^{\infty} E[X_j^2] < \infty$.

Now we borrow the trick from the lecture notes, as described in the hint. Suppose $\sum_{j=1}^{\infty} E[X_j^2] = \infty$ (so we will want to show a contradiction); then note that because $|X_j|$ almost surely we know that

$$\sum_{j=1}^{\infty} E[|X_j|^3] \leq \sum_{j=1}^{\infty} E[X_j^2]$$

Therefore, we can apply the CLT for non-identical variables (defining $D_n := \sum_{j=1}^{n} \mathrm{var}[X_j]$) and get

$$\frac{S_n}{\sqrt{D_n}} \to Z \sim N(0,1) \text{ in distribution}$$

But now let us fix some $t > 0$. Then

$$P[S_n > t] = P[S_n/\sqrt{D_n} > t/\sqrt{D_n}]$$

Since $t/\sqrt{D_n} \to 0$ and $S_n/\sqrt{D_n} \to Z$ (in distribution), we get that $P[S_n > t] \to 1/2$ as $n \to \infty$ for all $t > 0$, which means that S_n cannot converge in probability to any finite-valued random variable. This is a contradiction, and we're done.

Exercise 2. Let $\{X_n\}$ be a sequence of identically distributed random variables, with finite variance. Suppose that $\mathrm{cov}(X_i, X_j) \leq \alpha^{|i-j|}$, for every i and j, where $|\alpha| < 1$. Show that the sample mean $(X_1 + \cdots + X_n)/n$ converges to $\mathbb{E}[X_1]$, in probability.

Solution: Let

$$Y_n = \frac{\sum_{k=1}^{n} X_k}{n},$$

with $E[Y_n] = E[X_1]$. By Chebyshev's inequality,

$$\mathbb{P}\left(|Y_n - E[X_1]| \geq \varepsilon\right) = \mathbb{P}\left(|Y_n - E[Y_n]| \geq \varepsilon\right) \leq \frac{\mathrm{var}(Y_n)}{\varepsilon^2}.$$

The variance of Y_n is

$$\begin{aligned}
\mathrm{var}(Y_n) &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}(X_i, X_j) \leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha^{|i-j|} \leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha|^{|i-j|} \\
&\leq \frac{1}{n^2}(2n)\sum_{k=0}^{n-1}|\alpha|^k = \frac{2}{n}\frac{1-|\alpha|^n}{1-|\alpha|} \to 0.
\end{aligned}$$

Hence $Y_n \to E[X_1]$ in distribution, which implies (since $E[X_1]$ is a constant) that it converges in probability as well.

Exercise 3. Given an i.i.d. sequence $X_n, n \geq 1$ with $\sigma^2 \triangleq \mathrm{var}(X_1) < \infty$, the CLT states that

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\sum_{1\leq i\leq n} X_i - n\mathbb{E}[X_1]}{\sigma n^{\alpha}} \leq x \Big) = \Phi(x) \triangleq \int_{-\infty}^{x} \frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}} dt,$$

when $\alpha = 1/2$. Compute the limit above for every $\alpha > 0$ and every x.

Solution: Let

$$Y_n = \frac{1}{\sigma} \left(\sum_{i=1}^{n} X_n - n\mathbb{E}[X_1] \right).$$

For all x, n and α, we have that

$$\mathbb{P}\left(\frac{Y}{n^{\alpha}} \leq x \right) = \mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq x n^{\alpha - 1/2} \right) = \mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq \frac{x}{n^{1/2-\alpha}} \right)$$

There are two cases of interest

1. Let $0 < \alpha < 1/2$. Suppose $x \geq 0$. Then the sequence $\left\{ \frac{x}{n^{\frac{1}{2}-\alpha}} \right\}$ monotonically decreases to zero. Then for every ϵ there exists N such that for all $n > N$, $0 \leq \frac{x}{n^{1/2-\alpha}} \leq \epsilon$. By monotonicity of probability, for all $n > N$,

$$\mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq 0 \right) \leq \mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq \frac{x}{n^{1/2-\alpha}} \right) \leq \mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq \epsilon \right).$$

   Taking a liminf and a limsup and applying the CLT

$$\limsup_{n\to\infty} \mathbb{P}\left(\frac{Y}{n^{\alpha}} \leq x \right) \leq \lim_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq \varepsilon \right) = \Phi(\varepsilon)$$
$$\liminf_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{n^{\alpha}} \leq x \right) \geq \lim_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq 0 \right) = \Phi(0).$$

   Letting epsilon tend to zero, these bound coincide by continuity of Φ. Hence the limit exists and equals $\frac{1}{2}$. Similarly, for $x < 0$ the sequence $\{ \frac{x}{n^{\frac{1}{2}-\alpha}} \}$ monotonically increases to zero and yields the bounds

$$\limsup_{n\to\infty} \mathbb{P}\left(\frac{Y}{n^{\alpha}} \leq x \right) \leq \Phi(0) \qquad \liminf_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{n^{\alpha}} \leq x \right) \geq \Phi(-\varepsilon).$$

2. Let $\alpha > 1/2$. Suppose $x = 0$. Then, by the CLT

$$\mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq xn^{\alpha-1/2}\right) = \mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq 0\right) \to \Phi(0) = 1/2.$$

Suppose $x > 0$. Then the sequence $\left\{\frac{x}{n^{\frac{1}{2}-\alpha}}\right\}$ monotonically increases to infinity. Therefore, for every $M \in \mathbb{R}$ there exists $N > 0$ such that for all $n > N$ $xn^{\alpha-1/2} > M$ and

$$\mathbb{P}\left(\frac{Y}{n^{1/2}} \leq xn^{\alpha-1/2}\right) \geq \mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq M\right)$$

Thus

$$\liminf_{n\to\infty} \mathbb{P}\left(\frac{Y}{n^{\alpha}} \leq x\right) \geq \liminf_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{n^{\alpha}} \leq M\right) = \Phi(M)$$

and this bound holds in the limit. Therefore,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Y}{n^{\alpha}} \leq x\right) = 1.$$

Similarly, suppose $x < 0$. Then the sequence $\left\{\frac{x}{n^{\frac{1}{2}-\alpha}}\right\}$ monotonically decreases to infinity. Therefore, for every $M \in \mathbb{R}$ there exists $N > 0$ such that for all $n > N$ $xn^{\alpha-1/2} < M$ and

$$\mathbb{P}\left(\frac{Y}{n^{1/2}} \leq xn^{\alpha-1/2}\right) \leq \mathbb{P}\left(\frac{Y_n}{n^{1/2}} \leq M\right)$$

Thus

$$\limsup_{n\to\infty} \mathbb{P}\left(\frac{Y}{n^{\alpha}} \leq x\right) \leq \limsup_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{n^{\alpha}} \leq M\right) = \Phi(M)$$

and this bound holds in the limit. Therefore,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Y}{n^{\alpha}} \leq x\right) = 0.$$

In summary

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{n^{\alpha}} \leq x\right) = \begin{cases} \frac{1}{2} & 0 < \alpha < \frac{1}{2} \\ \Phi(x) & \alpha = \frac{1}{2} \\ 0 & \alpha > \frac{1}{2} \text{ and } x < 0 \\ \frac{1}{2} & \alpha > \frac{1}{2} \text{ and } x = 0 \\ 1 & \alpha > \frac{1}{2} \text{ and } x > 0 \end{cases}.$$

Exercise 4. Show that given an i.i.d. sequence $X_n, n \geq 1$ with mean μ, variance σ^2, while $(\sum_{1 \leq i \leq n} X_i - \mu n)/(\sqrt{n}\sigma) \to N(0,1)$ in distribution, it is not the case that the same sequence converges in probability. (Hint: Cauchy criterion)

Solution: By the Cauchy criterion if $Y_n \to Y$ in probability then

$$\mathbb{P}\left(|Y_{2n} - Y_n| \geq \varepsilon\right) \to 0,$$

and contrapositively, if $\mathbb{P}\left(|Y_{2n} - Y_n| \geq \varepsilon\right) \not\to 0$ then Y_n cannot converge in probability. Let

$$Z_n = \frac{1}{\sqrt{n}\sigma} \sum_{i=1}^{n} (X_i - \mu).$$

Observe that

$$\begin{aligned}
Z_{2n} - Z_n &= \frac{1}{\sqrt{2n}\sigma} \sum_{i=n+1}^{2n} (X_i - \mu) + \frac{1}{\sqrt{2n}\sigma} \sum_{i=1}^{n} (X_i - \mu) - \frac{1}{\sqrt{n}\sigma} \sum_{i=1}^{n} (X_i - \mu) \\
&= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{n}\sigma} \sum_{i=n+1}^{2n} (X_i - \mu) \right) + \left(\frac{1}{\sqrt{2}} - 1 \right) \left(\frac{1}{\sqrt{n}\sigma} \sum_{i=1}^{n} (X_i - \mu) \right)
\end{aligned}$$

and relabeling terms, as the X_k are i.i.d

$$Z_{2n} - Z_n = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{n}\sigma} \sum_{i=1}^{n} (X_i' - \mu) \right) + \left(\frac{1}{\sqrt{2}} - 1 \right) \left(\frac{1}{\sqrt{n}\sigma} \sum_{i=1}^{n} (X_i - \mu) \right),$$

where the $\{X_k\}$ and $\{X_k'\}$ are independent. By the CLT, the first term converges in distribution to an $N(0, \frac{1}{2})$ and the second term converges in distribution to an $N\left(0 \left(\frac{1}{\sqrt{2}} - 1\right)^2\right)$. Hence, $Z_{2n} - Z_n$ converges in distribution to an $N(0, \frac{1}{2} + \left(\frac{1}{2} - 1\right)^2)$, as these two terms are independent, and thusly cannot converge in probability to zero.

Exercise 5. Give an example of:

1. Independent zero-mean X_j's such that $\sum \mathrm{var} X_j$ diverges but

$$S_n = \sum_{k=1}^{n} X_j \tag{1}$$

   converges almost surely.

2. Independent zero-mean X_j taking values in $[-1, 1]$ such that $X_j \overset{\text{a.s.}}{\to} 0$ but S_n does not converge almost surely.

Solution:

1. Let
$$X_n = \begin{cases} 0 & \text{w.p. } 1 - \frac{1}{n^2} \\ \sqrt{n} & \text{w.p. } \frac{1}{2n^2} \\ -\sqrt{n} & \text{w.p. } \frac{1}{2n^2} \end{cases}.$$
Then
$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq 0) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$
As the X_n are independent, by the partial inverse to the Borel-Cantelli lemma, $\{X_n \neq 0\}$ only finitely often almost everywhere. Therefore, almost everywhere S_n is a sum of a finite number of terms and thus converges. Hence S_n converges almost everywhere. However,
$$\sum_{n=1}^{\infty} \text{var}(X_n) = \sum_{n=1}^{\infty} 2n \frac{1}{2n^2} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

2. Let
$$X_n = \begin{cases} \frac{1}{\sqrt{n}} & \text{w.p } \frac{1}{2} \\ -\frac{1}{\sqrt{n}} & \text{w.p } \frac{1}{2} \end{cases}.$$
Then
$$\sum_{n=1}^{\infty} E\left[|X_n|^4\right] = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$
Hence $X_n \to 0$ almost surely by lecture 18 proposition 1. However,
$$\sum_{n=1}^{\infty} \text{var}(X_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$
and thusly S_n cannot converge almost surely by theorem 2 of lecture 19.

Exercise 6. Let $\{X_n\}$ be a sequence of nonnegative integrable random variables and X an integrable random variable. Suppose $X_n \overset{\text{a.s.}}{\to} X$ and $\mathbb{E}[X_n] \to \mathbb{E}[X]$. Show that the family $\{X_n, n = 1, \ldots\}$ is uniformly integrable. Conclude that $X_n \overset{L_1}{\to} X$, i.e.
$$\mathbb{E}[|X_n - X|] \to 0$$
(Thus, $Y_n \overset{\text{a.s.}}{\to} Y$ is u.i. iff $\mathbb{E}[|Y_n|] \to \mathbb{E}[|Y|]$.)

Solution: As the X_n and consequently X are nonnegative the absolute values in the definitions of uniform integrability can be ignored. For all $b > 0$ with $\mathbb{P}_X(b) = 0$, $X_n \mathbb{1}\{X_n \leq b\} \to X\mathbb{1}\{X \leq b\}$ almost everywhere, as a probability measure can only have countable many items this condition can be ignored, i.e. a $b' > b$ always exists. By the bounded convergence theorem $E[X_n \mathbb{1}\{X_n \leq b\}] \to E[X\mathbb{1}\{X \leq b\}]$. Therefore, as $E[X_n] \to E[X]$, by linearity of integration $E[X_n \mathbb{1}\{X_n > b\}] \to E[X\mathbb{1}\{X > b\}]$.

By assumption all of the X_n and X are integrable. Thus they are individually uniformly integrable and any finite collection is also uniformly integrable, i.e. taking a max over a finite collection of integers. Let $\varepsilon > 0$. There exists $b_1 > 0$ such that $E[X\mathbb{1}\{X > b_1\}] < \varepsilon$. There exists an $N \in \mathbb{N}$ such that for all $n \geq N$

$$|E[X_n \mathbb{1}\{X_n > b_1\}] - E[X\mathbb{1}\{X > b_1\}]| < \varepsilon.$$

Moreover, as any finite collection is uniformly integrable, there exists a $b_2 > 0$ such that

$$\sup_{1 \leq n < N} E\left[X_n \mathbb{1}\{X_n > b_2\}\right] < \varepsilon.$$

Let $b \geq \max\{b_1, b_2\}$ then

$$\begin{aligned} \sup_n E\left[X_n \mathbb{1}\{X_n > b\}\right] &\leq \sup_{1 \leq n < N} E\left[X_n \mathbb{1}\{X_n > b\}\right] + \sup_{n \geq N} E\left[X_n \mathbb{1}\{X_n > b\}\right] \\ &< \varepsilon + E\left[X\mathbb{1}\{X > b_1\}\right] + \varepsilon \\ &< 3\varepsilon. \end{aligned}$$

Hence $\{X_n\}$ is uniformly integrable. As $X_n \to X$ almost everywhere implies $X_n \to X$ in probability, the result follows from theorem 1 of lecture 19.

Exercise 7. Let $N(\cdot)$ be a Poisson process with rate λ. Find the covariance of $N(s)$ and $N(t)$.

Solution: **Solution.** Suppose $s \leq t$. We have that

$$\begin{aligned} E[N(s)N(t)] &= E[N(s)(N(s) + (N(t) - N(s)))] \\ &= E[N(s)^2] + E[N(s)(N(t) - N(s))] \\ &= E[N(s)^2] + E[N(s)]E[(N(t) - N(s))] \\ &= \lambda s + \lambda^2 s^2 + \lambda s \lambda (t - s) \\ &= \lambda s + \lambda^2 st \end{aligned}$$

so that

$$\operatorname{cov}(N(s), N(t)) = E[N(s)N(t)] - E[N(s)]E[N(t)] = \lambda s + \lambda^2 st - \lambda s \lambda t = \lambda s.$$

Keeping in mind that we assumed $s \leq t$, we instead write this in the general case as

$$\operatorname{cov}(N(s), N(t)) = \lambda \min(s, t).$$

Exercise 8. Based on your understanding of the Poisson process, determine the numerical values of a and b in the following expression and explain your reasoning.

$$\int_t^\infty \frac{\lambda^5 \tau^4 e^{-\lambda\tau}}{4!}\, d\tau = \sum_{k=a}^{b} \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

Solution: The left-hand side is the probability that an Erlang random variable of order 5 and rate λ is larger than t, i.e., the probability of at most 4 arrivals over an interval of length t. The right-hand side is the probability that the number of arrivals in a Poisson process with rate λ, over an interval of length t, is between a and b (inclusive). Thus, $a = 0$ and $b = 4$.

Exercise 9. ***(practice problem, not for grade)***

(a) Shuttles depart from New York to Boston every hour on the hour. Passengers arrive according to a Poisson process of rate λ per hour. Find the expected number of passengers on a shuttle. (Ignore issues of limited seating.)

(b) Now, and for the rest of this problem, suppose that the shuttles are not operating on a deterministic schedule, but rather their interdeparture times are exponentially distributed with rate μ per hour, and independent of the process of passenger arrivals. Find the PMF of the number shuttle departures in one hour.

(c) Let us define an "event" in the terminal to be either the arrival of a passenger, or the departure of a shuttle. Find the expected number of "events" that occur in one hour.

(d) If a passenger arrives at the gate, and sees 2λ people waiting, find his/her expected time to wait until the next shuttle.

(e) Find the PMF of the number of people on a shuttle.

Solution: Solution:

(a) The number of people that arrive within an hour is Poisson-distributed with parameter λ, and its expected value is λ.

(b) If the interarrival times for the shuttles are exponentially distributed, then shuttle departures form a Poisson process of rate μ. Thus, the number of departures in one hour has a Poisson PMF with parameter μ.

(c) Here, we are merging two independent Poisson processes, which results in a Poisson process of rate $\mu+\lambda$. Therefore, the expected number of "events" occurring in one hour will be $\mu + \lambda$.

(d) The number of people waiting conveys some information on the time since the last departure. On the other hand, because of memorylessness of the exponential distribution, this number is independent from the time until the next departure. Thus, the expected waiting time is just $1/\mu$, irrespective of how many people are waiting.

(e) Every event at the airport has probability $\lambda/(\lambda + \mu)$ of being a passenger arrival ("failure") and probability $\mu/(\lambda + \mu)$ of being a shuttle departure ("success"). Furthermore, different events are independent. The number of passengers on a shuttle is the number of failures until the first success and is distributed as $K - 1$, where K is a geometric random variable with parameter $\mu/(\lambda+\mu)$. Thus, the PMF of the number of people on the shuttle is

$$\left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right), \qquad k = 0, 1, \ldots$$